DGK Deutsche Geodätische Kommission
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Enrico Mai

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Introduction

This work bibliographically reviews several aspects that relate to a new concept of geodesy, where the principles of relativity play a leading role. Ongoing improvements, especially in the development of optical atomic clocks and in laser technology, make it feasible to geodetically exploit relativistic effects instead of just correcting for it in highly accurate measurements. New observables, like direct observations of potential differences, become available and therefore require the development of new methods. Genuine geodetic tasks will certainly benefit from a consistent relativistic approach, and modern quantum sensor technology provides the practical means to further develop the emerging field of relativistic geodesy.

The theories of relativity (e.g. Misner et al. [366], Bergmann [39], Goenner [217], Stephani [524], Schutz [491]) and quantum mechanics (e.g. Kiefer [275]) have an increasing impact on geodesy. Apart from the relativistic modeling of space-geodetic techniques and an occasional use of relativistic concepts, geodetic practitioners in the most part still tend to the ideas of Newtonian mechanics and its classical concepts of space and time. With the advent of the era of artificial Earth satellites at the latest and its utilization in the framework of physical geodesy, it became obvious that relativistic effects no longer should be ignored (e.g. Eardley et al. [148], Huang et al. [251]). Novel measurement techniques for the determination of time and frequency using atomic clocks are also based on quantum-physical processes. Therefore, geodesy to some extent is directly affected by the on-going search in physics for a unified approach that might bring its both fundamental theories together (Audretsch [23]).

Regarding the exploitation of relativistic concepts, the Earth system's metric field will become a major subject of investigation in future geodesy. On the other hand, several newly available quantum sensors are potentially useful for genuine geodetic applications (e.g. Heitz [242], Börger [63]). Within the domain of physical geodesy, the utilization of highly precise time and frequency instrumentation, e.g., optical atomic clocks and frequency combs (Hänsch [231]), for the direct determination of the gravitational potential in combination with classical gravity field functionals is of special interest. Generally, different types of measurements are available based on the motion of bodies, light propagation, or clocks. Here we will focus on the last-mentioned option.

Relativistic geodesy (Müller et al. [380]), e.g., direct potential measurements via atomic clocks, in cooperation with the German national metrological institute PTB [433] shall be advanced and promoted, and thus it was one of the various research foci of the QUEST [437] cluster of excellence. Regarding applications, the global unification of national height systems (e.g. Colombo [105], Rummel [464]) is a suitable task. In geophysics, for the solution of the inverse problem, i.e., the detection of sub-surface density variations, the use of atomic clocks can help to reduce the existing degeneracy (Bondarescu et al. [59]). Basic formulae for the determination of heights using atomic clock readings, either by using frequency ratios or clock rate differences, are already given in literature (e.g. Brumberg/Groten [74]) and will be recapitulated herein in later sections.

The determination equation for the comparison of atomic clock readings depends on the relative state of motion of the involved clocks and its respective positions within a gravity potential (Moyer [370], [371]). In return, among others, these quantities could be determined by time and frequency comparisons between atomic clocks that are distributed in a (worldwide) network. For instance, geoid height determinations on the cm-level would require (optical) atomic clocks with an accuracy of about 10^{-18} (Herrmann/Lämmerzahl [246], Chou et al. [97]).

Before any problem-dependent exploitation of the theory of relativity, specific observation methods and measurement techniques had to be developed, which is not the intention of this review article. Instead, based on an extensive bibliography, it tries to pave an annotated way through the thicket of existing approaches. Their mathematical frameworks show a conflicting notation every now and then. In this respect, the text shall help the reader to identify its similarities and significant differences, respectively.

We first provide an introductory digest on several aspects of time. It intentionally comprises paragraphs on the meaning of time. An exclusive focus on practical results would be misleading because the nature of time is anything but evident. Consequently, its implications for possible applications, e.g., in the field of relativistic geodesy, seem to be far from being exhausted. Thus, special effort has been made to pick up some metaphysical aspects (Esfeld [172]), in an attempt to go slightly beyond the mathematical framework and technological issues themselves and into the context in which they belong.

Subsequent sections become more specific and outline rather practical issues that are related to an emerging relativistic, i.e. chronometric, geodesy. Following is a review of the already existing mathematical framework for relativistic approaches. Eventually, exemplarily calculations are given, regarding the above mentioned geodetic task that could possibly benefit the soonest from novel approaches based on foreseeable instrumentation. These computations may act as a starting point for more elaborate simulations, e.g., in preparation for chronometric leveling campaigns.

Time

Basically, time can be discussed on four different levels (Eisenhardt [166]), and intuitively we all know how to deal with time in everyday life. In scientific life however, especially in physics, we distinguish between three more elaborate varieties to be backed by theory: coordinate or parametric time (absolute), proper time (general relativistic) and a unified concept of time that still needs to be developed. The time specified by the one time coordinate of a four-dimensional space-time coordinate system is referred to as coordinate time, whereas in these terms proper time analogously refers to the spatial arc length in three-dimensional space. For the investigation of possible geodetic applications we will mainly focus on proper time τ and its relation to coordinate time t. In view of the reached state of presently available instrumentation (regarding precision, accuracy, and stability), these concepts are just fine. On the other hand, the more sensitive our devices get the more obvious the need of a consistent treatment of space and time concepts will become. The few next paragraphs provide a selective bibliographic review especially on various time aspects which, at a first glance, seem to be purely metaphysical. Nonetheless, they may prove to be very fruitful for the identification of starting points in future developments.

1 The search for the nature of time

Obviously, the nature of time remains a mystery (Davis [123]). The vast amount of literature at the boundary between philosophy and physics dealing with time (e.g., Whitrow [580], [581], Kroes [302], Aichelburg [3], Audretsch/Mainzer [24], Fraser [198], Filk/Giulini [185], Vaas [559], Eisenhardt [166], Petkov [418]) implies that there is no evident agreement on this topic.

An essay contest, initiated by the Foundational Questions Institute (FQXi [197]), once again revealed a wide range of ideas about time. Several authors reject the dominant role of time as being a redundant concept (Barbour [28], [29], Girelli et al. [212], Prati [428]) and replace it by a pure concept of relations (Rovelli [460]). Others insist on the unique features of time in comparison to space (Skow [507], Callender [83]) or even promote the idea of the existence of more than one time dimension (Weinstein [575]). Jammer [264] highlights the historical development of space concepts. As a distinct timewise property, the arrow of time and its implications are discussed (Carroll [90], Crooks [110], Ellis [170], Parikh [402], Vinson [566]). Describing time as a network and expanding an information space is just another way of introducing that time arrow (Halpern [229]). The position of ultrastructuralism (Rickles [453]) is not far away from this view. Considerations on causality (Liberati et al. [330]) and indeterminism are occasionally connected to the idea of a frozen arrow of time and finally even to the question of free-will existence (Stoica [529]). Remark: the whole concept of a time arrow itself is strongly connected to the individuals' ideas about space, it seemingly depends on their cultural background (Weiler [572]).

The problem of time is thought to be vanishing, if geometry isn't split from matter as it is done in Einstein's theory of general relativity, where geometry simply reacts to the presence of matter. Dreyer [146] therefore introduces an internal relativity concept. Another reservation about the concept of general relativity focusses on its alleged reduction of the time-parameter to a time-coordinate without physical significance which obscures the nature of time, meaning change (de Saint-Ours [132]). Of course, this alleged meaning is being contested too (Shoemaker [505]). There are also suggestions that the conceptual understanding of time and energy in Einstein's theory is still incomplete (Wiltshire [594]). The lack of a unified theory, i.e., a theory of quantum gravity (Callender/Huggett[82]), is said to be due to the different notions of time in general relativity and quantum mechanics (Gambini/Pullin [205]). Remark: Zych et al. [615] discuss a possibility to test the genuine general relativistic notion of proper time in quantum mechanics.

Consequently, the temporal character of quantum gravity becomes of interest (Kiefer [274]). To claim the problem of time simply being a paradox, the existence of space, geometry and gravity altogether may be rejected (Markopoulou [347]). On the contrary, there are arguments for the impossibility of the existence of global time (Minguzzi [364]). Time on one hand is thought to be created by measurements (Helfer [243]), but on the other hand there are cases against our ability to uncover the true origin of physical time by experiments at all (Jannes [266]). There even could exist theories (e.g. on time or on gravity) that are underdetermined by all actual and possible observations (Newton-Smith/Lukes [389]).

To enter a more detailed metaphysical discussion of time's nature is way beyond the scope of the present work. But in the long term, geodesy too has to be founded on a revised concept of time that is able to support a unification of the otherwise conflicting notions within the theories of relativity and quantum mechanics. In this respect, the space-time metric of a theory of quantum gravity will probably have to be dynamical simply because the concept of a static space-time seems not to be fundamental enough (Wheeler [578], [579]). There exist many alternative candidate theories in physics already.

Generally, a growth in fundamental validity of a theory on time is associated with growing structural poverty (Eisenhardt [166]). Dropping the distance measure, i.e., getting rid of the four-dimensional line element, transforms a metric manifold into a conformal manifold which raises the degree of freedom already. Subtracting angles too, leads to an affine manifold. If we also abandon the concept of parallelism, then we get a pure differentiable manifold. But still we can take measurements and attach real numbers to all of these manifolds. Next we could add fundamentality by foregoing coordinates and differentiability. Now we are left with only a topological space – neglecting neighborly relations would result in an ordered set at least. This seems to be the minimum ingredients for being able to create a unifying concept in physics. In case we drop ordering relations too, we end up with a pure set of unordered events which is not applicable anymore to our purposes.

General relativity, due to the lack of an external time, will not separate between different deformations of three-dimensional manifolds. It rather only distinguishes between different equivalence classes of points. Looking at the evolution of a system in the sense of detecting deformation changes, creates kind of a fundamental time. An open question in quantum theory is the emergence of locality from and within a fundamental global state. Basically, a unified concept of time will depend on the general character of the underlying physical theory. The more fundamental, i.e. formal, a theory is being laid out, the more timeless it will be. In contrast, theories of constructivism, e.g., Newton's theory of gravity or some candidates for a unified theory like the quantum loop theory for instance, will introduce time at the very beginning. Successful unification seems to require a combination of both, formalism and constructivism.

A 'theory of everything', being able to describe all four natural forces of physics within a single formalism, would be relevant also in practice, e.g. for geodesists, to take full advantage of modern day instrumentation like quantum sensors.

2 The measurement of time

Realization of a clock does not clear up the phenomenon of time, it rather defines a time scale in order to compare the occurrence of events. Besides periodical motions, radioactive decay processes may define characteristic time scales. Time in the sense of a coordinate time could only be measured by an idealized clock. This is quite a common view on time. Scientists often show a tendency to take a supposedly objective outside position in order to study a given system. Timing or pulsing means counting of certain recurring phenomena, e.g., phases or periods of an oscillatory motion. Harmonic oscillators are vibratory systems with ideal properties, and a clock simply is a device to create such a periodic process. In practice, we have to draw on real representatives of a clock, i.e., real world motions like pendulums, oscillatory crystals, planetary revolutions, atomic state superpositions, etc. Such devices, i.e. non-idealized clocks, do not provide coordinate time. On the other hand, the only tools needed to experimentally test whether a given clock is a standard clock (providing proper time) or not, are light rays and freely falling particles (Perlick [411]).

On a large astronomical scale, remote celestial objects such as pulsars (Hartnett/Luiten [235]) may also provide timing signals. The first pulsars were discovered in the late 1960's (Hewish et al. [248]). Regarded as highly precise time beacons, they are useful in different applications, e.g., tests on gravity (Stairs [519]) or navigation of vehicles in deep space (Sheikh et al. [503]). But one has to account for sudden mass redistributions that may result in erratic changes of the pulsar's rotational frequency. Furthermore, timing observations using astronomical objects require careful transformations between the terrestrial frame and proper frames of the astronomical object, taking into account a variety of relativistic effects (Hellings [244]). In general, the treatment of any (astrometric) observation comprises the solutions of the equations of motion representing the world lines of the observer (observing subject) and the observed object, and of the equations of light propagation, i.e. the light ray. Additionally, the observation process itself has to be modeled in a consistent relativistic framework.

There is no correct or incorrect way of choosing a process as basis for time measurements. Any choice is logically possible and finally leads to a consistent system of natural laws. But certain choices will enable much simpler physics than others. Nonetheless, the aspect of measurability is important because time is not identical to motion itself. Instead, the perception of time relies on the observable order of periodic motion. A good clock selection then requires a well-countable motion which is as regularly as possible. Of course, regularity is just another relative term, and thus, we simply compare periodic motions against each other. A pure comparison of two clocks will not reveal, which one is more accurate. There is kind of an inherent uncertainty about the degree of regularity of the last or best natural clock. Following the idea of the least common multiple, one could claim that, in principle, all observable oscillations are periodically equivalent and finally lead to the same time measure. To avoid circular reasoning, one would eventually have to introduce a timeless concept of (periodic) motion. As a remark, the definition of space exhibits the same difficulties. Today it is quite common to reject the idea of being able to measure absolute time and space at all. All we can possibly detect is a change in the relative positioning of material objects. There may exist only a series of stationary pictures that imply some kind of ordering. Thus, something non-spatial but non-structureless and something non-temporal but non-static would have to span our relational entities of space and time, respectively.

Measuring time (Audoin/Guinot [25]) requires some sort of a clock device. In general, the actual state of the universe restricts the number of possible ways to construct a functional clock (Eisenhardt [166]) by employing different types of available motion. The seemingly continuous expansion of the universe itself can be viewed as a time creating process. Today we witness a phase in the universe's evolution where matter is widespread distributed on different scales. The periodic motion of macroscopic objects can very well be used to construct mechanical clocks, e.g., on an astronomical scale by measuring the rotational or orbital motion of planets. Looking at past phases we realize that due to thermodynamic processes large scale structures did not yet exist. In such a regime only atomic oscillations could be used to design a clock. Going further backwards we note that even atoms were absent. Instead, only elementary particles and fields are at hand. Now, quantum mechanical principles had to be used. Constructing a clock becomes tricky because of the superposition of probability densities for isolated oscillations. There are no unique clock hand positions. Asymptotically approaching the initial state of the universe, space-time itself is now subject of quantum mechanical principles. Topology becomes variable, ordering starts to break up and space-time is no longer a smooth continuous entity. Any remaining selected oscillations grow faster and faster. The concept of a clock finally loses its sense. Back to the present state, we have a wide range of different technologies at hand to measure time (McCarthy/Seidelmann [351]).

3 The different notions of time

Parametric time t as a paradigm of classical, i.e. Newtonian, mechanics also enters quantum mechanics and even the concept of special relativity, where space-time is still treated as an invariant entity. Regarding time aspects in the combination of relativity with quantum theory, concepts of special relativity have significant consequences for particle physics (Giulini [213], sections 4.1-4.3). A special feature of quantum mechanics is the fact, that it operates with probabilities and allows for the superposition of system states before measurements actually take place. Time is read externally at these instances and is said to be static because it is independent of the inner motions of the quantum mechanical system and therefore measurable by fixed clocks in an idealized classical sense. In between of the measurements we assume an evolution of the system state's probability density. Consequently, the idea of indefiniteness had to be applied for space-time too. But the implications of superimposed space-times are anything but clear (Eisenhardt [166]). This problem still awaits a solution and should not be ignored, because macroscopic size of objects does not prevent us from applying quantum mechanics. Its validity is far from being limited to the (sub-)atomic scale. Moreover, quantum mechanics is the background theory of our atomic clocks, that we want to use for measurements of relativistic space-time events.

With the introduction of general relativity the character of time changes. Now, time dynamically depends on matter and/or energy and will retroact upon these entities. Time and motion are strongly coupled, and space-time itself becomes an object such that time no longer can be applied externally. Instead, we introduce the concept of proper time τ and therefore time changes from a global to a local entity with an intrinsic dynamism. Any motion bears its own, i.e. proper, time. Here, „local“ means uniform validity at any place at any time in the universe whereas „global“ applies to the universe as a whole. The structure of relativistic space-time is continuous and smooth, but it will possibly become ripped up if we want to allow for quantum mechanical ideas. This issue will be picked up again later when we mention the concept of a Planck scale.

4 The problem of a unified concept of time

Since decades, fundamental physics rests on two major theories, namely the principles of general relativity (e.g. Einstein [159], [162], Eddington [150], [151], Synge [536], Weinberg [573], Stephani [523], Ellis [168]) and quantum mechanics (e.g. Dirac [139], Trostel [553]), based upon different notions of time. Mainly due to this inconsistency, often called the problem of time, both theories could not be unified into a theory of quantum gravity, so far. At least, Dirac was able to compose a relativistic quantum mechanics for electrons (Kragh [300]). For details on the relativistic wave equation, consult for instance the introductory textbook by Stepanow [522]. Goenner [218] provides a historical retrospective of early attempts towards unification; see Barceló [30] for a detailed discussion of historic and contemporary theories of gravity and Kiefer/Weber [273] for some remarks on the interaction of gravity with quantum systems on a mesoscopic scale.

So far, any theory that is considered to be a candidate to achieve a unification tends to lead to tiny violations of at least one of the basic principles. For example, some theories modify the space-time frame in a way that violates the Lorentz invariance, other stipulate physical constants that evolve in time leading to violations of the equivalence principle or the local position invariance. Furthermore, there exist theories that modify Newton's universal law of gravity at certain scales or introduce new long range forces, e.g., by additional scalar fields.

The development of a theory of quantum gravity may require an extension of the relativity concept (Gefter [206]). In general relativity the focus is on the metric field using coordinate-like quantities (space and time), whereas quantum physics relies on Hamiltonian mechanics connecting coordinates with impulse-like quantities (energy and momentum). Applying the concept of relativity to both space-time and momentum space, eventually leads to the consequent use of the full phase space and the principle of relative locality (Amelino-Camelia et al. [12]), reviving an idea of Born [64]. As a consequence for practical applications, (atomic) clocks had to be supported by another measurement device, namely a calorimeter.

The persisting incompatibility (regarding the underlying concepts of time) between general relativity and quantum mechanics may possibly be overcome by a quantization of gravity. If matter is quantized then the corresponding fields should probably be quantized too. Otherwise we could end up with an infinite speed of light violating the indispensable causality principle (chapter 3.5 in Giulini [213]). In contrast to simultaneity, which always refers to a certain inertial system, causality can not become an observer dependent term. It has to remain an absolute entity. Two events are not simultaneous if they can causally interact on each other. Being an experimentally testable proposition, causal dependency will not travel faster than light.

Regarding the classical local realistic viewpoint (physical properties are defined before and independently of its measurements), which claims the existence of a few loopholes that allow for the rejection of the opposite quantum mechanics viewpoint (under certain conditions physical properties depend on its observation), it seemingly can be abandoned according to experimental observations (Giustina et al. [215]). Otherwise, one had to assume any secret form of superluminal information exchange. In this respect, a very careful revision of experimental setups and results is of course mandatory in order to independently verify any alleged violations of natural laws, as the example of neutrino detection experiments showed just recently (OPERA collaboration [399]).

Canonical quantum gravity, which is only one of various candidate theories, seems to require more than the four dimensions we got used to in order to describe space-time (Giulini [214]). Introducing extra dimensions will affect possible predictions on the variability of fundamental constants (Dent [130]).

5 The role of fundamental quantities and physical dimensions

The study of the constancy of fundamental quantities is strongly related to the realization of time scales. Besides its potential for becoming a distinct new device for geodetic measurements, atomic clocks are being used to reveal limits on the temporal variation of fundamental constants (e.g. Flambaum [188], Flambaum et al. [189]). As a remark, one usually implies a linear time dependence.

Any threshold corrections for presumed variations of dimensionless fundamental quantities on a large scale ultimately relate to variations of measurable quantities on a low scale. This relation can be constrained by experimental data resulting in allowed ranges for corresponding observables (Dine et al. [138]).

The important question of constancy and isotropy of fundamental constants is being heavily investigated. Evolving natural constants may provide valuable hints for a successful unification of the theories of general relativity and quantum mechanics (Peng [409]). The large numbers hypothesis (Dirac [141]) was one of the very first attempts to make a case for such an evolution.

In practice, radio astronomy experiments can be applied to study the constancy of the proton-to-electron mass ratio (Bagdonaite et al. [26]), linking the strong interaction (essential for the proton mass) and the weak interaction (essential for the electron mass).

Another example: the gravitational constant G is a fundamental parameter in celestial mechanics and satellite geodesy. It became a major subject of investigation in research on relativistic effects (e.g. Müller/Biskupek [379], Müller et al. [382]), too. Among others, a variable $G(t)$ would impact on the solution of fundamental problems in satellite orbit calculation, e.g. the classical Kepler problem (Schneider/Müller [483]). Some theories of gravity predict an anisotropy of G which would cause an additional solid Earth tide effect. As a test, resulting local gravitational accelerations could be measured by gravimeter measurements (Heitz [241]). Data analysis then allows to put limits on the amplitude of any such anisotropy and of any inequality between speed of light and speed of gravity (Will [584], Carlip [86], Schäfer/Brügmann [469]). According to the general theory of relativity, light and gravity travel at the same speed. Recently, an alleged experimental verification of this equality (Fomalont/Kopejkin [194]) gave rise to a controversy on this issue (e.g. Asada [18], Will [587], Samuel [466], Carlip [87], Kopejkin/Fomalont [296]).

Additionally, a possible anisotropy of the speed of light (Pelle et al. [408]) would violate the Lorentz invariance (Lämmerzahl [309], [310]) which itself is an essential element of special relativity (Lorentz et al. [333], Synge [535], Müller/Peters [375]). It guarantees that to each solution of the equations of motion in a stationary reference system there exists a corresponding solution for a reference system in motion. Within the mathematical section of this work, this correspondence motivates the concept of fictitious forces. By rotating two orthogonal standing-wave optical cavities and interrogating them by a laser, one could test and verify the Lorentz invariance up to the 10^{-17} level (Eisele et al. [165]).

Fundamental equation systems in physics, e.g. Maxwell's equations, are Lorentz invariant, i.e., its solutions obey the principles of length contraction and time dilation. Einstein [155], [156], [157], [158] mentioned immediate consequences of a variable speed of light (Ellis/Uzan [169]) by stating (Einstein [156]): „Beschränkt man sich auf ein Gebiet von konstantem Gravitationspotential, so werden die Naturgesetze von ausgezeichnet einfacher und invarianter Form, wenn man sie auf ein Raum-Zeitsystem derjenigen Mannigfaltigkeit bezieht, welche durch die Lorentztransformation mit konstantem c miteinander verknüpft sind. Beschränkt man sich nicht auf Gebiete von konstantem c, so wird die Mannigfaltigkeit der äquivalenten Systeme, sowie die Mannigfaltigkeit der die Naturgesetze ungeändert lassenden Transformationen eine größere werden, aber es werden dafür die Gesetze komplizierter werden."

Astrophysical phenomena like gamma ray bursts are being used to study the constraints of possible invariance violations (Laurent et al. [323]), but terrestrial experiments, e.g. clock comparisons, might be used as well (Mattingly [350]). By comparing the long-term readings of differently designed atomic clocks (Marion et al. [346]) one can identify stringent upper limits to possible variations of fundamental constants (Uzan [557], [558]). On the subtleties of dimensional considerations that may arise due to slightly different points of view in mathematics and physics, see for instance Wallot [570]. There is also a controversy on the actual number of fundamental constants (Okun [396], Duff et al. [147]).

In classical geodesy, different measurement devices are being used to determine essential geodetic quantities like directions, distances, heights or accelerations of gravity. Starting from these, other quantities can be derived, e.g. angles, areas, volumes, gravity gradients. Some of the derived quantities could also be measured directly by specific instruments. Each quantity has a physical dimension that is related to any of the seven basic units or a combination of these.

In practice, there might be interactions between the realizations of the basic units (Petley [419]), which is not a pettiness. Only the second, the kilogram and the kelvin are independently defined on the basis of local properties (Guinot [223]). But all basic units are also uniquely related to natural constants. Besides the kelvin, also the kilogram could be defined via a fixed value of Planck's constant (Robinson/Kibble [457]) instead of a material artefact (CGPM [95]). A redefinition of the kelvin may become available, if the Boltzmann constant can be determined more accurately in near future (Fellmuth et al. [181]).

Any laws of nature have to be expressed by fundamental constants that are dimensionless due to a suitable combination of quantities with certain physical dimensions. Otherwise a possible change in the realization of basic units could not be distinguished from a prospected long-term change in the alleged invariants. Dirac [140] put it concisely: „If anything is not dimensionless then you cannot discuss whether it changes or not."

6 The realization of time scales

In metrology, whose results are crucial to the successful implementation of a relativistic geodesy, fundamental constants play an especially important role (Karshenboĭm [268], Karshenboĭm/Peik [269]). Of all basic units (e.g. Mechtly [353], Thompson/Taylor [545]) the one that is most precisely defined and measurable is the second (Leschiutta [326]), which corresponds to the basic quantity time. This also affects the practical realization of other basic units, e.g. the meter (Quinn [440]), which corresponds to the basic quantity length.

Worldwide, several national laboratories are responsible for the realization of the basic units. Every laboratory at first establishes its own standards. In Germany, the PTB with its primary clocks contributes to the scale unit of international atomic time (Bauch [35]). The general question arises to what degree remote standards are equivalent (Willink [592]). As an example, the connection between two national laboratories is explained in Piester et al. [421]. McCarthy/Seidelmann [351] present actual procedures for the derivation and distribution of uniformly valid atomic time scales. Alternatively, one could carry the clocks between laboratory sites in order to compare laboratories' time scales. Accounting for general relativistic effects led to the introduction of dynamical time scales, cf. § 31.

The establishment of a consistent system of scales and units requires the unambiguous definition of terms. Consequently, relativistic effects were considered in the IERS conventions (e.g. IERS TN32 [256], IERS TN36 [257]). Post Newtonian level implementations of several resolutions can be found in literature (e.g. IERS TN29 [255], Soffel et al. [512]). Geodesy at all times had to incorporate and account for new definitions of time scales. Universal time is crucial for use in every day life on Earth. Its general relationship with ephemeris time and possible uses in astronomical applications are discussed in Mulholland [374]. Relativity considerations required the introduction of revised time scales (Seidelmann/Fukushima [495]) and its transformations (e.g. Thomas [543], Fairhead et al. [177], Fukushima [204]), which sometimes led to conflicting definitions (Fukushima et al. [203]).

7 The choice of an underlying theory and its impact on definitions

Despite the fact that Newton's classical mechanics was extended by both, the theory of relativity and quantum mechanics, it still serves as a reasonable approximation in case of moderate relative velocities, short distances, and weak gravitational potentials. Moreover, firstly one may feel more comfortable when sticking with the traditional definition of terms, its seemingly intuitive use and practical applicability.

According to Einstein's theory of special relativity, concepts like 'equal duration', 'equal (spatial) distance' or 'simultaneity of events at remote places' require a new stringent setting of rules to make real sense and a corresponding instrumentation to measure it. Terms like 'uniform' and 'straight', as being used in the Newtonian laws of motion, become relative: uniform with respect to which clock, straight with respect to which system of reference ? The required number of rules depends on the nature of the quantity in question.

Carnap [88] distinguishes between so-called extensive and intensive quantities. The former require 3 rules, namely defining the concepts of equality, unit, and combinability. Extensive quantities can be combined via operations, either additively (e.g. weight, length, velocity in Newtonian physics) or non-additively (e.g. trigonometric functions, velocity in (special) relativistic physics). On the other hand, intensive quantities require 5 rules to measure it. Besides defining the terms of unit and equality, one has to specify the concepts of inequality, origin (i.e. zero point or reference point), and finally the scale. As an example, measurements of the material temperature and hardness or tone pitches can not simply be added. More precisely, also distances can not be added arithmetically. Instead, we add numbers that represent these distances. The latter are just configurations in physical space that could be combined. Combining or adding in a physical sense is quite different to a simple addition in a mathematical, i.e. arithmetically, sense. The measurement of several physical quantities relates to measurements of spatial distance, i.e. length, and temporal duration. Consequently, length and duration can be regarded as primary quantities.

Direct measurements deal with quantities whose values can be expressed by rational numbers. Irrational numbers are being introduced by theory in order to formulate physical laws and perform calculations. Observations provide no exclusive constraint for us to decide whether values have to be expressed by rational or irrational numbers. It's more a matter of convenience to either use a discrete or continuous number scale for expressing a physical law.

8 The backing of a theory by experiments

In practice, there exist different methods to measure a certain physical quantity, e.g. spatial distance or length. Consequently, one could introduce different definitions of the term length, which of course is quite impractical. Instead of defining physical terms by a complete set of operational rules, one should rather treat them as theoretical terms that become more and more specific in the course of development. Therefore, quantitative terms like 'length' are not explicitly defined, but remain theoretical. The best we can get from theory, i.e., physical postulates and operational rules, is a partial interpretation of the term in question. In this sense we can keep a single meaning of 'length'. Different measurement methods will simply draw connections between terms of a theoretical language and the language of observations. Expressing a physical law by means of quantitative terms is much easier and shorter than doing it qualitatively. Even more important, quantitative laws are readily applicable.

The deflection of light phenomenon could well be expressed under retention of Euclidean geometry. In fact, we have to choose between several alternatives of formulation. The proper choice is not a question of wrong or false theory but convention and suitability. This does, of course, not mean that there are no rivalling non-equivalent theories. Theories comprise conventions and natural laws alike. A single separated theory could not be examined at all. Empirical tests are itself interlaced with theoretical terms. In practice, it may be difficult to set up comparative experiments which allow us to rule out a theory or not. Theories are said to be equivalent, if they lead to the same predictions in all cases. Newton's and Einstein's theories of gravitation both make testable predictions (Schutz [490]). The differences in these predictions are so tiny in many cases, that it requires very sophisticated experiments and highly precise measurements to decide, which theory provides the more accurate predictions. In the very end, successful theories are being selected by evidence.

Theoretical predictions relate to observable effects. One can distinguish universal effects which are substance-independent (e.g. contraction of a rod due to gravity) from so-called differential effects which depend on the subject's material (e.g. contraction of a rod due to temperature). In contrast to differential effects, universal effects can entirely be eliminated from physical laws by an according reformulation of the underlying theory. Every time a universal effect and a corresponding law stating the conditions for the appearance and the magnitude of this effect show up in a certain system, the physical theory could be reformulated such that the magnitude of the effect becomes zero. As a result, the final theory will become simpler. Without these simplifications the question on the structure of space can not be answered in a clear way. Universal forces can always be interpreted as geometrical properties of space-time (Filk/Giulini [185]). Only combinations of geometry and universal forces may be tested by experiments.

Laws of nature have to be justified by experience (Einstein [163]). Following the theory of relativity in an a-priori sense, (mathematical) geometry tells us nothing about reality. There is no proposition that connects logical certainty with information about the geometrical structure of the physical world. The emphasis of the importance of physical geometry was crucial for the invention of the theory of relativity. Without this new interpretation, the transition to uniformly valid covariant equations probably would never have been occurred.

In case of competing theories, i.e., systems of physical laws, the general question arises on how to find a criterion to judge about differences in its predictive power with regard to observable events. The cognitive content of a physical law depends on its applicability for making predictions. In order to test these predictions, we will make a draft on many observations, not only a single one. In this context, a physical law simply rests upon the description of a multiple observed orderliness, i.e. regularity. In contrast, any single observed irregularity in the future significantly breaking the empirical laws will instantly rule out a theory. As a remark, Lorentz' ether theory is still of the same quality as Einstein's special relativity theory regarding observable predictions (Filk/Giulini [185]). Following the reasoning of Ernst Mach, i.e., his demand to consider only observable entities, one could reject Lorentz' theory because of its assumption of an ether which itself turns out to be unobservable. For the same reason we would prefer the concept of an observable relative or intrinsic time instead of an unobservable absolute or extrinsic time. As a consequence, simultaneity becomes a relative term, depending on the observers' state of motion.

Depending on the magnitude of predicted effects, evolving accurate measurements will continually tighten the limits of the theories' validity. It strongly depends on the available instruments whether a quantity or predicted effect can actually be treated as an observable or not. Consequently, there is also no sharp line between empirical and theoretical laws. Once non-observable entities might become observables if new technology enters the arena. The advent of highly precise optical clocks and time transfer methods is just an example.

9 The different kinds of geometry

Following Einstein's theory of relativity, space has a structure which in gravitational fields departs from the structure of Euclidean geometry. However, if the gravity field is weak, these deviations are difficult to observe. In Euclidean language one could say that, for instance, rods in a gravitational field are subject to contraction, whereas in non-Euclidean language (Reichenbach [450]) the laws of mechanics and optics remain the same as in pre-Einsteinian physics. That means, light rays in vacuum are straight lines which are not bent or deflected by gravitational fields. The behavior of rods can be expressed in both ways. Keeping Euclidean geometry leads to new physical laws.

We are free to choose any kind of geometry for the physical space we like, as long as we are willing to accept necessary corrections to the physical laws both in mechanics and in optics. A physicist should make his decision for a certain geometry before he chooses his measurement method for lengths. These choices should be consistent. Once the measurement method is fixed, the structure of space becomes an empirical question which can be answered by observation.

Regarding relativity theory, most physicists choose non-Euclidean geometry (Bonola [60]) whose language is more complicated than in Euclidean geometry but avoids the introduction of much more complicated physical laws to be consistent with observations. Only by the introduction of a joined concept of space-time we can keep an invariant distance measure. Allowing for a non-Euclidean geometry is just the price we have to pay for this convenient invariance (Jammer [264]).

10 The fundamental role of the line element

Four-dimensional world lines, i.e. geodesics, can be described mathematically in a simple way with help of a special metric that was introduced by Minkowski. Today we call a flat space-time a Minkowski space. Explicitly expressing the differential line element makes the diverse characters of space and time more apparent.

We start with the usual spatial element of arc in cartesian notation

$$\mathrm{d}r^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\,, \tag{1}$$

and introduce a time coordinate in spatial dimension by making use of the finite speed of light c

$$\mathrm{d}r = c\,\mathrm{d}t \quad \rightarrow \quad \mathrm{d}r^2 = \left(c\,\mathrm{d}t\right)^2. \tag{2}$$

The combination of the spatial and temporal coordinates results in an invariant spatiotemporal line element (Sexl/Schmidt [497])

$$\mathrm{d}s^2 = \left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) - \left(c\,\mathrm{d}t\right)^2. \tag{3}$$

The mathematical part of this work will provide a more detailed discussion of the line element, because the appropriate use of a metric, i.e., the formulation of the invariance of space-time intervals, is essential for the successful application of relativistic geodesy. Here we notice that the definition of space and time affects three fundamental concepts that are of great importance to (physical and relativistic) geodesy, namely the clock hypothesis (clocks measure proper time), the length hypothesis (whether or not to introduce a fundamental length to fix the scale), and the concept of a field quantity. Regarding the latter issue, depending on the actual metric coefficients, i.e. the line element, we may distinguish between different special cases of a field. For instance, in case of time-independent coefficients we call it stationary. If there are no time-space cross-terms than we speak of a static field, which means that all motions of any particles and fields are time reversible (Harwit [236]). These distinctions are directly related to the above mentioned arrow of time problem.

The choice of a specific metric, i.e., the formulation of the invariance of space-time intervals, is eventually based upon an empirical basis. Einstein [163] gives some additional remarks on our ability to imagine and visualize non-Euclidean geometry.

The minus sign in equation (3) epitomizes the different character of space and time. The dimension of time is geometrically mapped onto a spatial axis. In principle, any spatial reference system and time scale is possible. Regarding space, one often introduces (quasi-)inertial systems, but non-inertial systems are by no means unphysical or forbidden (Giulini [213]). It is just a question of convenience, because non-inertial motion of a reference system causes some extra terms describing apparent forces, e.g. Coriolis force, within the equations of motion. Special relativity theory distinguishes between real and apparent forces and therefore admits a special role to inertial systems. In opposition to classical mechanics, where the equations of motion are invariant against Galilei transformations, in special relativity these equations shall be invariant also against the more general Lorentz transformations. Going one step further by geometrizing gravity, the theory of general relativity abandons the existence of distinguished reference systems altogether. In this theory, mass densities and pressure terms or mass/energy fluxes are regarded as sources of gravitational fields. The fundamental difference between apparent or inertia forces and gravitational forces vanishes, at least locally. Both can formally be described by a single gravitational field, but now being dynamic in space and time.

The minus sign further implies that space-time can not be split up in an entirely arbitrary way. Time is separating and joining events while maintaining a consistent causal ordering. In principle, depending on the actual chosen splitting, space will change in time and vice versa. This kind of arbitrariness, as a key property of the theory of relativity, eliminates the existence of a single time. Instead it allows for an infinite number of equally valid time conventions.

Minkowski diagrams (Minkowski [365]) are often used to illustrate the principles of (special) relativity and constant speed of light (Mermin [359]). Applying it on the laws of motion of light rays and celestial bodies, e.g. planets, one recognizes that the path of a world line in any gravitational field is a geodesic. A unified theory of geodesy strongly relates to the general interpretation of geodesy problems as geodesic flows (You [611]).

The theory of general relativity renounces the concept of (a gravitational) force. A planet orbits the Sun not because of the Sun attracting the planet by a gravitational pull. Instead, the Sun's mass causes a negative curvature of the non-Euclidean space-time. The straightest possible world line of a planet, namely its geodesic, then equals the path of its actual motion around the Sun. An elliptic orbit, e.g. in case of an idealized two body problem, is not a geodesic in three-dimensional space. Rather, the planets' world line in four-dimensional non-Euclidean space-time is a geodesic. The same holds true for the path of a single light ray or photon. The concept of gravity as a force is replaced physically by the geometric structure of a four-dimensional system of space-time.

One should not confuse mathematical and physical geometry. The former is purely logical, whereas the latter is an empirical theory. Einstein transformed one physical theory (of gravity) into another. It is still physical, because non-mathematical terms are included, e.g., the distribution of curvature in space-time. Playing an important role in cosmology (Berry [40], [41], Weinberg [574], Lambourne [306]), different views on the curvature of space (Friedman [202]) led to a debate (Realdi/Peruzzi [447], Realdi [448]) on whether the world as a four-dimensional manifold would be cylindrical (Einstein [161]) or spherical (de Sitter [133], [134], [135], [136]). Following that debate, many alternative cosmological models were developed till this day.

Relativistic Effects

According to Kurt Lewins' „nothing is so practical as a good theory", senseful application of a theory requires prior testing. Both versions of the theory of relativity have been and are still being tested in various ways. Another reason for testing is the hope for getting some hints in the search for alternative physical theories. These alternatives then may become more promising candidates for a 'grand unified theory'. This section gives some remarks on selected prominent individual tests of relativistic effects and its implications. Furthermore, it highlights a few existing proposals for future testing.

11 Testing the concept of relativity

Many different theories have already been developed to account for the idea of relativity. To falsify irrelevant candidates, several parameterized theoretical frames have been introduced. Some of these focus solely on special relativity (Mansouri/Sexl [342], [343], [344], Vargas [563], Günther [226], Sfarti [499]). A more flexible frame, namely the parameterized post-Newtonian formalism (Will [586]), is surprisingly suitable (Will [590]) for discussions on the validity of relativistic metric theories of gravitation. Unfortunately, it tacitly assumes the local validity of special relativity already. It is based on the idea that the metric tensor in the near zone of a source of gravity can be expanded into a series in inverse powers of the fundamental speed, namely the speed of light in vacuum (Kopejkin et al. [299]), forming the primary parameter. In the past, due to limited experimental capabilities, the deemed independence of special and general relativity tests was reasonable. But with a tremendous increase in measurement precision, the need for a consistent frame setting becomes obvious. Therefore, Tourrenc et al. [552] discuss the possibility of new relativity tests by means of a generalized approach comprising a blend of appropriate theoretical frames.

12 Focussing on Einstein's theory of relativity

Tailored tests of special relativity (Giulini [213]) mainly focus on the validity of the relativity principle itself, properties of the speed of light c, and its practical implications. According to the relativity principle, physical laws are equally valid in all inertial systems. In the sense of Newton, this principle is restricted to classical mechanics. Einstein extended its scope of application to electrodynamics, and therefore to the propagation of electromagnetic waves, e.g. light rays. He combined this general principle with some special properties of light. According to his theory, c should be of limited constant value, isotropic, and independent of the relative velocity of the light source.

In special relativity inertial systems are still marked reference systems. Changing from one inertial system to another, only the light's frequency (Doppler effect) and propagation direction (aberration effect) may change, its speed remains the same and isotropic. The finite speed of light poses a limit for the transmission of information and action/energy. Signals can not be transmitted instantly. Nevertheless, looking at visual phenomena in the propagation of light waves, phase velocities and group velocities may be of larger value than c in accordance to special relativity. The same holds true for the cosmological escape velocities of far remote galaxies (Filk/Giulini [185]). Furthermore, among any set of inertial systems there exist no prominent ones. These fundamental statements lead to a number of derivable effects like time dilation, length contraction, relativistic Doppler shift and aberration (Stumpff [530]), that all have to be accounted for in the interpretation of highly precise observations. The mathematical section of this work provides the fundamental formulas of the most significant relativistic effects.

Only decades after the presentation of special relativity, several visual phenomena resulting from length contraction got a proper explanation. Basically, one has to distinguish between sensed geometry and measured geometry of a body, simultaneous location of its parts, and simultaneous perception, respectively.

The experimental foundations of general relativity (Thorne/Will [547], Will [582], [583]) rest upon a series of famous experiments. As more precise technology becomes available, refined relativistic tests are performed or repeated. Inspired by the historic experiment of Hafele/Keating [227], [228], a renovated experimental confirmation of relativistic effects (position in gravity field, state of motion, Sagnac effect) via flying atomic clocks was achieved by Davis/Steele [122], for example. As a remark: these effects are routinely corrected for in space geodetic techniques, e.g., in the operation of the Global Positioning System (GPS) (Spilker [518], Eardley et al. [148]), otherwise its positioning results would lead to errors in the order of a few kilometers.

13 Testing relativity via earthbound and space-bound experiments

Traditionally, one distinguishes between three classical tests (Sexl/Sexl [498]), already suggested by Einstein, and subsequent modern tests. Those classical tests specifically pertain to predictions on the deflection of light by the Sun, the gravitational frequency shift of light, and the anomalous precessional shift of Mercury's perihelion. The last phenomenon is historically reviewed in detail by Roseveare [458]. Surprisingly, Einstein was not aware of other important consequences of his own theory, e.g. the gravitational time delay, i.e. Shapiro delay. In case of the Sun, this effect is superimposed by another time delay effect due to the Sun's coronal electron plasma (Muhleman/Johnston [373]). Relativistic tests have to account for a number of superimposed effects, which places high demands on the experiments' design and execution. The Shapiro effect (Shapiro et al. [501]) may be measured between drag free spacecraft (Ashby/Bender [22]), where non-gravitational disturbing effects will be shielded. The drag-free satellite control (Theil [541], Fichter et al. [182]), after some preliminary conceptual considerations (e.g. Lange [319], [320]), led to prototype space vehicles (e.g. DeBra et al. [124]) and eventually became a standard feature, especially in geodetic satellite missions, e.g. GOCE (Rummel et al. [463], Yi [610]).

Lämmerzahl [313] highlights underlying assumptions in the relativistic theory of gravity. One key element is the equivalence principle (Einstein [160]), first termed this way in Einstein [156]. In Einstein's theory of relativity, the weak equivalence principle, i.e., the proportionality of inertial and gravitational mass, is a prerequisite for the possible geometrization of gravitation. If this principle were violated, as predicted by some alternative theories of gravity, the gravitational force could not be reinterpreted as curvature of space-time, as being done by Einstein. The Eötvös experiment (Dicke [137]) was one way to test this proportionality. Earthbound experiments, e.g., the Pound-Rebka experiment (Pound/Rebka [427]) for testing the gravitational redshift, are exposed to a series of perturbations. The success of this experiment relied on the exploitation of the Mössbauer effect (Wegener [571]). A gravitational redshift occurs because light looses energy as it escapes gravitational fields. This loss is proportional to the field's strength and results in a longer wavelength and lower frequency, respectively. However, the effect is very small and superimposed by another redshift phenomena due to relative motion and the expansion of the universe. It is a real challenge to experimentally separate those effects.

When performed in space, corresponding free-fall experiments are much less susceptible to several noise sources. Various microgravity environments are technically available: vacuum drop towers, atmospheric parabolic flights, sounding rockets, and (drag-free controlled) orbital platforms. The last-mentioned alternative is superior if long duration free-fall is required. On the other hand, satellite based experiments are by far the most demanding option in terms of costs and development time.

There are proposals for space missions comprising atomic clocks to accurately measure the gravitational frequency shift (Schiller et al. [473]). Likewise, the sensitivity of equivalence principle tests improves by several orders of magnitude (Worden [604]). The remaining disturbances are mainly due to residual gravity gradients, tracking errors and gas pressure effects.

Space conditions show several advantages in comparison to earthbound experiments (Lämmerzahl/Dittus [311]). Experiments in space (Dittus et al. [142]) are indispensable if we want to test predictions of low-magnitude relativistic effects. Bertolami et al. [42] discuss the discovery potential of gravitational experiments and existing alternative theories of gravitation. As an example, the gravitomagnetic field may be detected by modern gravity gradiometer experiments (Paik [400]). There was a debate on what experimental technique is more sensitive to gravitomagnetic interaction, either lunar laser ranging (LLR) (Murphy et al. [385]) or specialized space probe missions (Kopejkin [297]). There is plenty of literature available on the relation between LLR and tests on gravity (e.g. Merkowitz [358]), e.g. gravitomagnetism (Soffel et al. [513]). Thanks to improved instrumentation, lunar laser ranging with an accuracy on the 1-mm-level is in reach (Kopejkin et al. [298]). Within relativistic geodesy, LLR continues to be valuable tool (Müller et al. [378], Biskupek/Müller [44], Müller et al. [381]).

One of the amazing consequences of gravitomagnetic fields, which can be illustrated by massive rotating bodies and the finite speed of propagation of its gravitational attraction, is the so-called frame dragging, or Lense-Thirring effect (Thirring [542], Lense/Thirring [325], Mashhoon et al. [349], Lämmerzahl/Neugebauer [308]). Measuring it (Pfister [420]), e.g. via satellite laser ranging (Ciufolini [99], Ciufolini et al. [101], Iorio [262]), results in a test of Mach's principle (Brans [66], Misner et al. [366], Narlikar [387]), too. Technology improvements enable alternative experimental settings which meet high demands on the precision of measurements. A test of the Coriolis effect was already suggested in the late 1950's (Pugh [434]). Gyroscopic tests of Einstein's theory of general relativity using spacecraft were envisaged with the advent of the first artificial satellites (Schiff [471], [472]). As an example, the Gravity Probe B mission (Everitt [174], Everitt et al. [175], Will [591]), initially proposed in the early 1960's (Fairbank/Schiff [176], Cannon Jr. et al. [85]), was designed as an Earth orbiting laboratory. Further improvements in the experimental setting could lead to potentially even more sensitive gyroscopic missions (Lange [321]).

14 Alternative modeling of gravitation

Tests of relativity can not only be performed in earthbound laboratories (Chen/Cook [96]) or in Earth's orbit, but also in the solar system and beyond (Turyshev [555], Damour [120]). Interplanetary space probes proved to be useful in relativity tests, e.g. the Viking spacecraft (Reasenberg et al. [449]). They are especially useful in the search for occasionally proposed long distance violations of Newton's law of gravitation. The equivalence principle is subject to investigation for both, laboratory test masses and massive bodies (Shapiro et al. [500]). Lämmerzahl [307] proposed a generalized weak equivalence principle in order to cover quantum phenomena too. This may pave the way for a better understanding of the nature of gravity (Lämmerzahl [314]).

Some alternative theories predict a violation of the strong version, i.e., the existence of an external field effect (Blanchet/Novak [56]). Observational data of the spiral arm rotation of galaxies indicate a violation of the Newtonian law of dynamics. Various gravitational theories do exist. For example, modified Newtonian dynamics (MOND) (Milgrom [362], Blanchet [54], Blanchet/Le Tiec [55]) and its relativistic upgrade (Bekenstein [38]) were set up to circumvent the unpleasant dark matter hypothesis. There exist severe doubts, based upon recent astronomical observations (Moni Bidin et al. [368]), on proposed dark matter models at least in case of the solar neighborhood. The MOND claims a nonlinear relation between force and acceleration when gravitational acceleration is extremely low, which had to result in a long-range modification of Newton's law of gravitation.

The space age revealed several anomalies, e.g., the Pioneer anomaly (Jaekel/Reynaud [263], Johann et al. [267], Turyshev/Toth [556]), some of which still await an explanation (Preuss et al. [430]). Quantitatively, an assumptive cosmological expansion of the universe most likely does not explain these anomalies (Carrera/Giulini [89]). For a discussion of lastingly unexplained phenomena in celestial mechanics and astronomy we refer to Lämmerzahl et al. [312].

15 Progression of the interferometric method for relativity testing

Interferometry is a well-established metrological method. It either uses light waves (optical interferometers), or matter waves (atom interferometers) (Bouyer et al. [65], Cronin et al. [109]), that are being superimposed to extract information, e.g., on the rotational status of the interferometer itself. The wave nature of matter can well be observed and studied in earthbound drop tower experiments (van Zoest et al. [562], Müntinga et al. [384]).

Applying matter waves, atom interferometry (Sterr/Riehle [527]) becomes an alternative to experiments comprising classical test masses (e.g. Hubler et al. [252], Kleinevoß et al. [281], Nolting et al. [393], Schwarz et al. [492], Luo et al. [336]). It might also be used to test the invariance of the Newtonian gravitational constant (Fixler et al. [187], Lamporesi et al. [315]). Milyukov et al. [363] provide a review of rather traditional (e.g. torsion or free fall) experiments on the determination of G. Some proposals for future space missions comprise the combined use of both, microscopic and macroscopic test masses in order to investigate possible violations of established laws of gravitation on different scales (Amelino-Camelia et al. [11]). Atomic spectroscopy can improve the limits on possible variations of other fundamental quantity too, e.g. the fine structure constant α (Fortier et al. [196]). A variable α would affect the readings of atomic clocks and hence its comparability. In gradiometry, instead of (possibly miniaturized) extended test-masses used up in conventional accelerometers, quantum engineering technology employs individual atoms or atomic clouds as identical drag free test masses (Maleki et al. [340]).

Earthbound atom interferometers, in principle, can be regarded as atom gravimeters (Peters [414], Schilling et al. [474]) or devices that also allow for the determination of the Sagnac effect (Post [426], Riehle et al. [454], Schneider [482]). In general, they can be used to measure external fields, e.g. inertial forces. Steffen et al. [521] suggest an instrument based on single trapped atoms being able to measure potential gradients with a precision of $5 \cdot 10^{-4}$ in units of gravity acceleration g and, with the implementation of further technological improvements, eventually rivaling in precision free-fall atom interferometers. Among others, atom interferometers provide the means to test different theories of gravitation by experiments on Earth. Theories make different predictions, e.g., on the universality of free fall or the universality of clock rates, i.e., the gravitational redshift. It is questionable whether (Müller et al. [376]) or not (Wolf et al. [600], [601]) the latter effect is testable by atom gravimeters (Sinha/Samuel [506]). Applying different atomic ensembles enables the concept of differential atom interferometry (Eckert et al. [149]). Various quantum sensors are currently under development (Gilowski/Rasel [211]).

Another testable prediction of Einstein's theory of gravitation is the existence of gravitational waves. Earthbound gravitational wave detectors could be significantly improved (Punturo/Lück [436]), e.g., by highly stabilized lasers (Willke et al. [593]) or by using non-classical light sources (Vahlbruch et al. [560]), that are mostly based on parametric processes in non-linear optical materials. Freise et al. [200] discuss the implications of different topologies for gravitational wave detectors. Both, laser interferometry and matter wave interferometry, can be applied for the purpose of detection (Delva/Rasel [129]). If interferometry is based on light waves, then laser astrometric tests of relativity (Turyshev et al. [554]) could be performed. In principle, a constellation of at least three formation flying spacecrafts, e.g., as within the upcoming Laser Interferometer Space Antenna (LISA) mission (Rüdiger et al. [461]), enables time delay interferometry (Tinto [549]). By now gravitational waves could not be detected.

16 Clocks as relativistic sensors

In general, relativity has to be exploited in two alternative ways. Single time measurements make use of clock readings, and clock comparisons additionally make use of electromagnetic signals. Both means are affected by relativistic effects, the causes of which can be described in terms that are of interest in geodetic applications.

The availability of precise atomic clocks make the theory of relativity a practical tool, i.e., we can regard and apply those clocks as relativistic sensors. The theory of relativity, in its special (Einstein [154], Giulini [213], Will [588]) or general form (Einstein [159], [162], Will [586], [589]), is a key element in modern metrology (e.g. Maleki/Prestage [339], Guinot [224]).

Special relativity states that a clock's ticking rate in comparison to another ones' depends on their relative states of motion. Likewise, following general relativity, clock readings are affected by the clocks' position within a gravitational potential. Or, to be more precise, following Filk/Giulini [185] „the world line (of a clock) in a gravity field is shorter".

Conversely, velocities and potential values can be determined by clock comparisons. Starting from these primary quantities, secondary quantities may be derived, e.g. potential differences or heights. All these quantities are relative ones. Therefore, as in classical geodesy, the question of reference frames (Soffel et al. [509]) is essential for relativistic geodesy.

Electromagnetic signals traveling through space will be affected in various ways (frequency shift, time delay, path deflection) due to several properties of disturbing massive bodies (mass, spin, irregular shape). Reversely, highly precise measurements of these effects on a signal, in principle, allow for the determination of the massive objects' properties (Ciufolini/Ricci [100]).

Some fundamental features of the theory of relativity, e.g. the finite speed of light, have to be accounted for even in earthbound classic geodetic instrumentation, e.g. absolute gravimeters (Nagornyi [386]), if high-precision measurements are to be taken.

In the early days of relativistic testing, experimental settings relied on comparatively high relative velocities (Reinhardt [451]) and big changes in elevation, i.e. changes of the gravity potential. Thus, the theory of relativity was first respected on astronomical scales (e.g. Clemence [102]). Planetary space missions are also very well suited for testing modern gravitational theories (e.g. Anderson et al. [15]). With the growing precision and stability of frequency standards (Guinot/Arias [225]) it became apparent that the annual motion of the Earth about the Sun imposes significant variations in the readings of clocks that are attached to Earth's surface (Clemence/Szebehely [103]). Later, observational equations accounted for many different physical causes of these variations (Moyer [370], [371]).

Today, with the advent of 10^{-17} to 10^{-18} (frequency inaccuracy level, $\delta f/f$) frequency standards, relativistic effects can even be detected in experiments that make use of only moderate to small earthbound velocities (Chou et al. [97]) and changes in position (Pavlis/Weiss [403]). In experimental physics, applications of the attosecond ($1as = 10^{-18}s$) (Wengenmayr [576]) are already in reach (e.g. Kienberger/Krausz [277], Corkum/Krausz [107], Krausz/Ivanov [301]). On the other hand, relativistic geodesy (Müller et al. [380]) nowadays still owns the status of a somewhat experimental science. Proving its applicability to contemporary open questions (e.g. a unified world geodetic height system), the beneficial use of relativistic geodetic techniques should become evident.

17 Apparent limits on the resolution

Possibly all physical quantities, including time and space, are finally discrete. 'Hodon' and 'chronon' are suggested terms for the minimal values of length and time, respectively. Consequently, discrete time would consist of tiny leaps and in between two steps there could not run any physical processes. In engineering practice however, space and time are still regarded as being continuous. From a mathematical point of view this is important, because otherwise the traditional concept of a limit value, which is a prerequisite in physics for introducing velocities or accelerations etc., would become inapplicable.

Measurements of space and time can not be performed with an arbitrary precision or on an arbitrary scale. Ultimate limitations seem to be imposed on one hand by the existence of a Planck scale (Filk/Giulini [185]), where fundamental quantities such as length, time, mass, temperature or charge, and derived quantities exhibit certain extremal values that are interconnected by natural constants, e.g., speed of light c, gravitational constant G, Planck's constant h, or the reduced Planck's constant $\hbar = h/2\pi$, respectively. For length, time, and mass we find the minimal values

$$
\begin{aligned}
l_P &= \sqrt{\frac{\hbar G}{c^3}} && \approx 1.6 \cdot 10^{-35}\, m \quad \text{(Planck length)}, \\
t_P &= \sqrt{\frac{\hbar G}{c^5}} = \frac{l_P}{c} && \approx 5.4 \cdot 10^{-44}\, s \quad \text{(Planck time)}, \\
m_P &= \sqrt{\frac{\hbar c}{G}} && \approx 2.2 \cdot 10^{-8} kg \quad \text{(Planck mass)}.
\end{aligned} \tag{4}
$$

The nominal values for l_P and t_P impose lower limits for any localization in space and time (Giulini [213]).

As a consequence (Eisenhardt [166]), the resolution of distance measurements via light rays is theoretically limited by the light's wavelength λ. The latter is related to frequency f and the speed of light ($c = \lambda f$). From elementary physics we know that frequency depends on energy via Planck's constant ($E = hf$). So, a minimum energy level is associated with the existence of smallest possible distances: $E_{\min} \propto \lambda(f)$. On the other hand, following the theory of general relativity, an object of mass m will curve space-time with the so-called Schwarzschild radius $r_S = 2Gm/c^2$ as a limit for the radius of curvature ρ. For $r < r_S$ the escape velocity value would have to exceed the speed of light, which already gives a hint on the existence of black holes. This is just an apparent 'coordinate singularity'. Only at $r = 0$ we face a true singularity (Filk/Giulini [185]). Due to the equivalence of mass and energy $E_0 = mc^2$ (Okun [397], [398]), we find another limit for the measurement of small distances: $E_{\max} \propto \rho(m)$.

Transition to relativistic geodesy

Already in classical geodesy one tries to relate essential measurements directly to time. For instance, distances correspond to time measurements by determining the round trip time of an electromagnetic signal between distant points. As another example, gravity can be measured by the determination of the time period of a pendulum's oscillation. It is a fundamental feature of classical geodesy to apply concepts like 'reference direction' and 'reference plane', 'simultaneity of events' (Jammer [265]), 'Earth rotation' and so on in a Newtonian sense. Newtonian mechanics assumes that space and time are both absolute. As we now know, this is only a first approximation to reality, because time and spatial motion are not decoupled phenomena.

To start with the transition from classical to relativistic geodesy we will treat space-time as an absolute entity, i.e. sharply defined, herein. For the solution of earthbound geodetic problems, especially since the advent of artificial Earth satellites and its use within space geodetic techniques, relativistic effects became non-negligible (e.g. Ashby [20]). For the ease of practical work, the task of taking relativistic measurements should not be superimposed with the task of pursuing relativistic tests alongside. Several measurement campaigns and satellite missions are dedicated to the determination of refined confidence intervals for various parameters of candidate theories of gravitation.

Instead, depending on the attainment of a chosen adequate precision level of our instrumentation, for the time being, we take for granted the validity of Einstein's theory of relativity. Still being the most reliable relativistic extension of the theory of gravity, it should simply be regarded as a new tool (Will [585]). Many different post-Newtonian approximations were examined, but Einstein's theory of gravity remains the simplest alternative, which so far passed all tests regarding its predictions with great success. Remark: recently, Kiefer et al. [276] confirmed the existence of Einstein's „relativistic mirror“.

Applying Einstein's general theory of relativity, we assume that the metric tensor is the only gravitational field variable (Kopejkin [299]). Nonetheless, this theory has its own limitations, which acts as a motivation to develop more elaborate concepts of gravity in fundamental physics. The successful exploitation of any theory of relativity requires a consistent relativistic modeling of observations (Klioner [289], [291]).

In physics, one is more interested in pure metrological applications of atomic clock readings, whereas in geodesy we focus on various traditional tasks, e.g., the determination of potential differences, that might benefit from this new type of direct observations.

18 Selected technological issues

The relativistic framework of actual instrumentation is presented in mathematical detail in Bordé [61]. Moreover, modern day measurement technologies make use of quantum-physical processes. Frequency and time measurements with atomic clocks are based upon certain state transitions (Lombardi [332]). Gill [210] describes several suitable atomic transitions for optical frequency standards in detail.

Precisely speaking, there is a difference between a pure frequency standard (Riehle [455]) and a clock. The latter is more demanding from an engineering point of view. Frequency standards periodically generate pulses at a rate as regular as possible. It may take some time for the equipment to settle in order to achieve a required level of regularity. Making it a clock requires the additional implementation of a counting unit, which continually records the number of pulses. For practical application, it should be possible to easily turn on and off a clock. Another desirable property would be the mobility of the whole device.

The quality of a frequency standard can be characterized by certain quantities: accuracy, precision, and stability (McCarthy/Seidelmann [351]). Depending on a proposed application, for the selection of a suitable clock, one has to find a compromise, because high accuracies require comparatively long averaging times, whereas stability naturally decreases with time. Relativistic geodesy, in view of the typical time scales involved within the geophysical processes of the system Earth, requires high accuracy and stability at the same time.

Precise frequency standards are often still based on atomic hydrogen maser oscillators (Vessot [565]) and atomic fountain clocks (Wynards/Weyers [606]). Microwave clocks are progressively being replaced by atomic clocks that rely on so-called optical transitions. These new frequency standards operate on 4 to 5 magnitudes higher frequencies than microwave clocks. The development of optical clocks (Peik/Sterr [407]) required the availability of ultra-stable interrogation lasers (Sterr et al. [526]). Additional improvements in laser technology, e.g., the realization of so-called superradiant lasers (Bohnet et al. [58]) with linewidths of a few mHZ, may lead to even more stable atomic clocks (Meiser et al. [355]).

Optical lattices (Takamoto et al. [537]) are created to trap countable (in a statistically sense) sets of individual particles (atoms or ions) (Katori et al. [270]), the interrogations of which are suitable for constructing precise frequency standards. In case of neutral atoms one typically traps 10^3 to 10^4 particles, whereas ion-based devices trap only a few particles (or even a single ion). The trapping is necessary to eliminate the (classical) Doppler effect. Generally speaking, the interrogation of trapped particles is suitable for constructing a variety of measurement devices. The application of quantum logic in precision spectroscopy indicated the feasibility to construct optical clocks based on single ions (Schmidt et al. [478]). As another example, single electrons may be used as a magnetometer (Hanneke et al. [230]), e.g., for the determination of the fine structure constant α.

The performance of a frequency standard is degraded by various sources of (residual) frequency shifts and noise. Besides relativistic effects, various physical processes may lead to frequency shifts in an (optical) atomic clock, such as electric fields, magnetic fields and atom collisions. Much effort is spent on the shielding of the clock instrumentation and the reduction of the individual sources of noise, but eventually the intended application of the clock determines the distinction of signal and noise. To increase the performance of a clock, physicists have recourse to various measures, e.g., operating the apparatus in a vacuum regime, optimizing the interrogation technique, using comparatively long averaging times (measurement duration) of up to several hours, etc.

The dependency of the relative frequency instability on the averaging time is usually expressed by the Allan variance (e.g. Allan [6], Schlüter [475], Peik/Bauch [406]). It is still one of the most prominent quality measures of clock performance or frequency stability (Barnes et al. [32], [33], Riley [456], Allan [9]) because of the divergent behavior of the classical variance for some correlated time series (Allan [8]). One of the best clock performances in terms of inaccuracy so far was confirmed by the NIST laboratory using trapped ion clocks based on Al^+ $(9 \cdot 10^{-18})$ and Hg^+ $(2 \cdot 10^{-17})$, respectively. Remark: the same measure (Allan variance/deviation) is used to characterize the quality of different techniques for remote time and frequency comparisons (Piester/Schnatz [422]).

The level of frequency shifts (Zeeman effect, Stark effect, cold collisions, spectral purity, leakage, neighboring transitions, etc.) can be reduced, for instance, by using higher transition frequencies, which is a motivation to replace microwave clocks by optical clocks. Ultra-high performance can be achieved by reducing perturbation effects as far as possible. In order to reduce thermal noise, the isolated atoms or ions are put nearly at perfect rest by means of laser cooling techniques (Schmidt [477]). This improves the signal-to-noise ratio and even more so the overall quality of the clock. Reaching the cooling limits takes a certain amount of time, where this procedure is part of a comparatively long total preparation process which is necessary for the entire laboratory instrumentation setup. In this respect, ion clocks are a bit easier to handle.

Wineland et al. [595] discuss how to achieve a requested measurement precision within a reasonable time frame. The measurement duration clearly restricts possible geodetic and metrological applications. Advanced interrogation techniques, e.g., based on correlation spectroscopy, enabling shorter averaging times of a few minutes (Chou et al. [98]), will support applications that require a higher temporal resolution of clock readings.

Another approach to further improve systematic clock shift compensation and suppression uses certain optical nuclear transitions by utilizing a specific electronic level in both the nuclear ground and isomer manifolds. Outstripping even the performance of conventional optical atomic clocks, nuclear clocks based on so-called virtual clock transitions composed of stretched states within suitable electronic ground levels could result in clock operation with total fractional inaccuracy approaching $O(10^{-19})$, or even $O(10^{-20})$ (Campbell et al. [84]). As a reminder, a fractional frequency shift of $1 \cdot 10^{-19}$ corresponds to a $1\,mm$ discrepancy in clock height.

The clock performance can also be increased by another technological improvements. For instance, a better laser beam guidance (Kleine Büning et al. 2010) enables longer interrogation times and possibly even continuous laser interferometric measurements. In quantum optics, the ability to create coherent laser light with high frequency stability is a prerequisite for precise measurements (Lisdat/Tamm [331]). Femtosecond laser frequency combs (Hollberg et al. [250]) show great potential for the development of new instrumentation. They are already used, for instance, for the transfer of stability. Lezius et al. [329] discuss how highly accurate long distance measurements based on spectral interferometry with frequency combs could be performed. Femtosecond laser based space metrology is already in reach (Klein/Bedrich [279]). An alternative to atomic clocks and gravimeters (using a free falling atomic sample) is based on the levitation of a Bose-Einstein condensate (BEC) which shows enhanced measurement sensitivity (Impens/Bordé [261]).

Remark: (optical) cavity cooling (Wolke et al. [602]) seems to be superior to the laser cooling approach, but rather for the purpose of matter wave generation for atom lasers and BEC studies. Regarding geodetic applications, it may enable more accurate measurements of rotation and gravity accelerations (atom interferometry, atom gravimetry). In principle, matter waves may also enable a (Compton) clock design, directly demonstrating the connection between time and mass (Lan et al. [316]).

Another promising advancement in the transition of atomic clocks towards a practical geodetic instrument comes from successful miniaturization attempts. Frequency reference devices with a total volume of only a few cubic centimeters with reasonable precision and stability seem to be technically feasible (Kitching et al. [278]).

19 Clock networks requiring time and frequency transfer

In the following, unless stated otherwise, 'clock' is meant to be an atomic clock, based on either microwave or optical transitions. Basically, atomic clocks give proper times. The difference between proper time scales and coordinate time scales, as well as its connection to atomic clock readings is being discussed in detail by several authors (e.g. Aoki [16], Winkler/Van Flandern [596], Guinot [222], Arias [17]) and it is outlined in later sections on the mathematical framework of relativistic geodesy.

In view of geodetic applications, we tend to prefer relative measurements of time, instead of 'absolute' measurements at a single location. Consequently, we have to compare clock readings in a suitable way. Ashby/Allan [19] present theoretical and practical aspects on how to set up a coordinate clock network. Exploiting the capabilities of highly precise clocks requires the ability to intercompare frequency standards at the same level (Abbas et al. [1]). Therefore, synchronization of a clock network is a major issue, e.g., for combined measurement campaigns within networks of globally distributed observation sites. Without synchronization clocks could not be successfully distributed, neither on Earth nor in space, in order to globally read time and attain a causally consistent temporal order of remote events. The general network topology for different synchronization methods is explained in Wing [603], whereas Bregni [67] provides more technical details.

20 Time and frequency transfer via clock transportation

Synchronization (Klioner [285]) or syntonization (Wolf/Petit [598]) of different clocks in a relativistic sense can practically be achieved by various means. Clocks are either located nearby at the same site or they are separately located at remote sites. In the first case, clock readings may easily be compared in a direct way. In case of remote clocks, one could bring them together in one location by physical transportation. This requires portable clocks in the sense that they are moveable.

Following the theory of relativity, clocks in motion potentially bear information on its state of motion and the environment along its path of transportation. To gather and possibly exploit this information, we need the clock(s) to be continuously ticking during transportation phases. In case of a continuously operating clock during transportation, we call it a mobile clock. To this date, from a technological or engineering point of view, this requirement is much more challenging than simple movability, where the clockwork can be switched off and on occasionally. Clock synchronization by (slow) transport is equivalent to Einstein's rule for synchronization.

There already exist several suggestions for specific measurement procedures in case one can use mobile clocks in order to compare and apply its readings at remote stations (Bjerhammar [47]). Today, mobile high-performance optical clocks are under development. Regarding its performance, at the moment, the (Strontium based) non-mobile optical (lattice) clock at PTB is one of the world's best optical clocks, not least due to an highly accurate determination of room temperature induced frequency shifts (Middelmann et al. [361]). It has an accuracy of about $3 \cdot 10^{-17}$ and a stability of about $5 \cdot 10^{-17}$ for an averaging time of $\tau = 100\ s$, under controlled laboratory conditions. The quality measures of transportable counterparts are still worse by a few orders of magnitude.

Furthermore, till this day, aside from validating earthbound or airborne experiments (e.g., round-the-world flying clocks aboard a plane) to dedicated space missions concerning certain predictions of the theory of relativity, there is only limited practical experience with mobile frequency standards on a routinely field work basis, e.g., with a passive hydrogen maser (Feldmann [180]).

21 Time and frequency transfer via signal transmission

Alternatively to clock transport, one can follow indirect approaches by exchanging information between sites. Quantum mechanic entanglement of macroscopic objects has already been demonstrated (Schnabel et al. [479]), and also the transition of quantum information into materials via the interaction of light and matter is under investigation (e.g. Stute et al. [531], [532]). Nonetheless, using such techniques for operational information exchange over very long distances referring to clock readings will probably remain fiction for quite some time. A working option is the classical exchange of (electromagnetic) signals. Perlick [412] discusses geometrical subtleties of clock synchronization based on this method. From a mathematical point of view, relativistic formulae for time and frequency transfer up to order c^{-3} have been derived (e.g. Blanchet et al. [51]).

The transfer medium could either be 'free air' or 'by wire' (Piester/Schnatz [422]). The former option means going through the atmosphere with all its complicated environmental effects on the electromagnetic signal. Environmental influences on signals are much more manageable, if we use a material connection, e.g. glass fiber, between clock sites. Optical fiber networks for time and frequency transfer (Foreman et al. [195]) and dissemination (Amemiya et al. [13]) have already been used to achieve an optical frequency transfer with 10^{-19} relative accuracy over a distance of more than 100 kilometers (Grosche et al. [219], Terra et al. [539]). Other experiments demonstrated the feasibility of optical clock comparisons using telecommunication fiber links for extended distances up to about 900 kilometers (Schnatz et al. [480], Predehl et al. [429]).

If (very) long distances have to be bridged, e.g., on an intercontinental or global scale, then fiber networks are of limited use. Besides technological issues, its cost efficiency probably loses against free air transmission techniques. If there is no direct line of sight between remote clock sites, we may set up a signal chain. Depending on the wavelength of the electromagnetic signal, one could use atmospheric layers to bounce off and forward the signal. Accurate transfer methods require shorter wavelengths and therefore the use of artificial reflectors. Aiming at highest accuracies for very large distances, laser ranging techniques using two-way signals require active or at least passive transponders instead of reflectors (Degnan [125]). Depending on the actual application and its constraints, reflectors or transponders could be mounted on balloons, air vessels, stratosphere planes, satellites or even the Moon and other celestial bodies (e.g., if interplanetary distances have to be bridged).

Lasting problems in transferring time or frequency over large distances (which requires a precise delay determination etc.) is one of the most limiting factors in real clock applications like chronometric leveling, especially if sub-centimeter accuracies shall be achieved.

22 Time and frequency transfer methods

Depending on the sites' network geometry and available instrumentation, several time and frequency transfer methods (through the atmosphere) have been established. These can be classified using different categories. For instance, there are single view or common view methods (Schmidt/Miranian [476]), and one-way or two-way methods (Petit/Wolf [416]), respectively. Stable clock readings require comparatively long averaging times. This poses some topological constraints on the use of moving reflectors or transponders. In this respect, using a geostationary orbit would be of great benefit, but probably there will be limited access to commercial satellites for scientific time transfer studies, if any. On the other hand, regarding routinely realizations of time scales like TAI and UTC (cf. § 31), corresponding comparisons between national metrological institutes have been conducted for years now with leased satellite transponders, e.g., via Intelsat's IS-3R satellite (Zhang et al. [612]).

Several dedicated space missions exist (ongoing and/or planned) that either make use of comparatively low orbital heights (signal comparison/exchange experiments) or highly elliptical orbits (relativity testing), where the latter configuration is chosen in order to exploit larger differences in terms of gravitational potential and velocity along the trajectory of the instrumental platform that allow for an easier detection of various effects.

Despite the fact that two-way methods are of advantage, e.g., for the elimination of unwanted systematic effects, one-way methods are most commonly used. Several active artificial satellites carry passive retro-reflectors onboard, mainly for the reason of precise orbit determination. These reflectors can readily be applied for time transfer and dissemination. Besides positioning, this is one of the main purposes of global navigation satellite systems (GNSS) like GPS (American), GLONASS (Russian) or GALILEO (European).

As long as the two way satellite time and frequency transfer (TWSTFT) approach (Hanson [232]) is more expensive, the application of GNSS (Levine [327]) with its established geodetic phase and code measurements is of advantage (Ray/Senior [446]). For example, the GPS time signal (Klepczynski et al. [282]) is exploited by customized receivers, and one way GPS carrier phase time transfer (Delporte et al. [128]) has been used in the past to study the isotropy of the speed of light (Wolf/Petit [599]). Recent advances in GPS based time and frequency comparisons are discussed in Feldmann [180]. Other satellite constellations are equally useful, e.g., GLONASS (Lewandowski et al. [328]) or PRARE (Bedrich/Hahn [37]).

For highly precise signal transfers, connections to the satellites should be established via stable microwave or laser links (e.g. Klioner/Fukushima [287], Fridelance et al. [201], Petit/Wolf [417]). As an example, within the Jason2 satellite mission an optical link (time transfer by laser link [T2L2]) was established in order to calibrate the onboard oscillator with respect to ground clocks which led to improved altimetry and positioning results. Estimating the standard uncertainty in frequency transfer (Douglas/Boulanger [144]) is essential. Kleppner [284] remarks that, at high levels of precision, uncontrollable fluctuations might act upon atomic clocks such that it will be impossible to select a master clock for keeping true time. Furthermore, for ground-to-space time transfers with picosecond-accuracy via laser link, as with the upcoming ISS on-board package 'Atomic Clock Ensemble in Space' (ACES) (Spallicci [516], Salomon et al. [465], Cacciapuoti et al. [80], Švehla [534], Daganzo et al. [113], Heß et al. [247]), the signal detection process has to factor in the optical to electrical detection delay within the instrumentation setup (Prochazka et al. [431]). One aims at a few picoseconds time stability for the comparison of distant clocks and about one hundred picoseconds time accuracy for the time scale distribution.

Obviously, to accomplish highly precise time transfers, many effects have to be taken into account. One way of checking the achieved performance is a loop-wise use of multiple transfers, similar to the leveling method in classical geodesy. By doing so, one could study resulting non-zero closures, e.g., for triplets of two way satellite time and frequency transfers (Bauch et al. [36]).

Geodetic use of atomic clocks

In traditional physical geodesy, one of the main tasks is to establish and monitor a global geodetic observing system (GGOS, Plag/Pearlman [424]) which comprises three major parts, namely the geometry, rotation, and gravity field or geoid of the Earth. Geodesists contribute to the detection of the actual state of all three parts and its changes. Due to the combination of different effects, a functional GGOS requires an accuracy level for its individual observation techniques and reference systems of at least 10^{-9}. Regarding Earth dimensions with radius $R_\oplus \approx 6378\,km$ and a gravity value at the surface of about $g_\oplus \approx 9.81\,m/s^2$, an order of 10^{-9} means that we want to determine metric quantities on the millimeter level and gravitational quantities on the microgal level. Present gravity models (e.g. EGM08) and (quasi-)geoid models (e.g. EGG08) both have an accuracy on the decimeter level. Current gravity field space missions like GOCE aim at a geoid accuracy on the centimeter level and a gravity anomaly field on the milligal level with a spatial resolution of about $100\,km$ (Pail et al. [401]). Typical reference stations of the international reference system of gravity (epoch year 1971) have an accuracy of approximately $100\,\mu gal$, whereas modern classical gravimeters can reach the μgal (absolute gravimeter) or even sub-μgal (superconducting gravimeter) level (Torge/Müller [551]).

23 Decorrelation of physical effects by means of clock readings

Changes in the system Earth involve several interacting spheres, namely the biosphere, geosphere, atmosphere, cryosphere, and hydrosphere. There exist models to all parts of the system, the parameters of which shall be monitored, resulting in corresponding time series. Changes in form of variations, deformations, and rotations are induced by a variety of correlated causes. We are able to separate these to a certain extent, because they show a broad spectrum in amplitude and frequency. Spatially, we can distinguish between local and global effects. All these space-wise and time-wise differences imply corresponding requirements on geodetic instrumentation for observations. Consequently, the characteristic properties or specifications of individual clocks as a measurement device may be sufficient or not for a given task.

Different time scales (here in a non-technical sense of time intervals) are involved, which has consequences on the necessary frequency stability of the clock. Short intervals are associated with signal travel times or epoch distances. Intermediate intervals are needed for comparisons of atomic time scales (related to elementary particles, i.e. micro-scale objects like electrons or photons) with ephemeris or astronomical time scales (related to celestial bodies, i.e. macro-scale objects, like Earth, Moon, Sun). Long intervals are associated with large-scale Earth system processes, e.g. plate tectonics. Basically, Earth's gravity field connects different geo-disciplines. It therefore plays a dominant role in Earth system research. The combined analysis of globally distributed accurate clock readings can help to separate and study Earth system changes.

Atomic clocks are operated in vacuum. Nonetheless, in practice, any clock reading is affected by several effects (state of motion, gravity field, magnetic field, any possibly imperfect shielding from environment, etc.). Its output, which is kind of a proper time, can be written as a functional of different parameters describing these effects (e.g. Moyer [370], [371]). In principle, problem-dependent observation equations (regarding a specific effect as a perturbation or observation) can be set up to determine those parameters. In order to separate effects, differently or equally constructed clocks at various locations in space-time might have to be operated.

24 From theoretical relativistic framework to real world scenarios

Given general relativistic formulas on gravitational physics (e.g. Misner et al. [366]), it is not trivial to derive special cases for real-world applications. Already the Newtonian limit (Ehlers [153], Lottermoser [334]) of a three-dimensional rotating perfect fluid in equilibrium is quite complicated in its analytical formulation, even more so if multipole moments (potential coefficients) are not to be neglected (Kopejkin [294]).

The Newtonian limit is characterized by three requirements. First, any particles/bodies, apart from photons, move slowly with respect to the speed of light c, such that we can apply the following special case for the equation of a geodesic (for mathematical details we refer to § 29.1, especially equation (14), § 29.5, and equation (50)):

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau} \quad \Rightarrow \quad \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} + \Gamma^i_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0. \tag{5}$$

Furthermore, we stipulate weak gravitational fields which can be considered as perturbations of a flat space, i.e., one can decompose the general metric g_{ij} (cf. equation (17)) into a Minkowski form η_{ij} (cf. equation (46)) plus a small perturbation h_{ij} with $\|h_{ij}\| \ll 1$. Finally, the Newtonian limit only considers static gravitational fields (unchanging with time with respect to a (quasi-)inertial system).

For real-world applications tidal phenomena and dissipative forces play an essential role that can not be approximated by a simplifying perfect fluid assumption. Alternatively, one can try to treat such elaborate problems iteratively (Lottermoser [335]) or numerically (Meinel et al. [354]). Accounting for precession and nutation in a consistent relativistic manner enables the construction of a numerical theory of Earth rotation using the post-Newtonian model of rigidly rotating multipole moments (Klioner et al. [290], [292], [293]). In general, the comprehensive relativistic formulation of rotation remains a trouble spot. Usually, the relativistic notion of rotation (Moritz/Hofmann-Wellenhof [369]) either uses the inertial compass (gyroscope axes, local criteria) or the stellar compass (incident light rays of remote stars, non-local criteria) (Weyl [577]). Additionally, the concept of an ephemeris compass exists, where a dynamical reference system is defined by the motion of non-remote (in comparison to quasars) celestial bodies, e.g., planetary and/or satellite orbits. The concept of rotation in general relativity, its effects and measurement techniques using interferometry is discussed in Feiler et al. [179].

Using a canonical dynamics approach, one can rigorously show that Newtonian gravity is well embedded into general relativity and results from it in the weak-field slow-motion limit (Schäfer [470]). Depending on the actual gravity regime in place, i.e., its weakness or strongness, the same observed phenomena may be used to probe or test alternative theories of gravity, see Psaltis [432] for a discussion of this distinction between probing and testing.

Calculations of the time delay and frequency shift of light are mostly based on the integration of the null geodesic equations, but there exist alternatives. If 10^{-18} precision is requested then relativistic terms up to order c^{-4} have to be taken into account. Teyssandier et al. [540] present parameterized formulae for a specific case, where an isolated axisymmetric body causing a weak static gravitational field (due to its total mass and quadrupole moment J_2) slowly rotates with a certain internal angular momentum (Thompson [544]). Xu et al. [609] provide a unified formula approach in case certain different effects in combination affect atomic clock reading comparisons. It also enables the consideration of coupling terms.

Applying the theory of relativity to a given system mathematically means to solve a set of field equations, as we will see later on. Einstein's system of ten equations implies that everywhere a certain combination of space-time curvatures has to equal a corresponding combination of energy or energy flux of matter. Given a certain mass distribution, we can search for an equivalent geometry of space-time (Filk/Giulini [185]). Basically, one solves for the metric (field), which allows for the calculation of the corresponding gravitational field which in turn enables a direct determination of resulting trajectories of test masses. In sum, it is far from being a trivial task.

The first exact solutions of Einstein's field equations were found for highly simplified problems, e.g., a single mass point (Schwarzschild [493]) or an incompressible liquid sphere (Schwarzschild [494]). Several authors list known exact solutions for more general problems (e.g. Stephani et al. [525]). Occasionally, new ideas appear to potentially ease the mathematical treatment of general relativity, e.g., Penrose [410] replaced the usual tensor approach by a spinor formalism. In principle, there are formally correct solutions to Einstein's equations that lead to logical contradictions, e.g., violations of the causality principle. These difficulties will be resolved or vanish, because we also have to take additional equations into account, describing matter itself. The solution to the latter equations rules out all cases that are inconsistent with the geometry of space-time (Filk/Giulini [185]). In combination, all formulae form a set of deterministic equations.

25 The resurrection of the chronometric leveling idea

Accurate optical clocks have the potential to be exploited in several areas (e.g. Soffel [511], Peik/Bauch [406]). As an example, the idea of a chronometric leveling already came up decades ago (Vermeer [564], Schüler [488]) by reasoning that clock rates are shifted by changing gravitational potentials. Differences in potential values relate with differences in height. As in the classical approach, relativistic height determination is inevitably tied to the definition of reference surfaces. Correspondingly, one can introduce the concept of a relativistic geoid (Kopejkin et al. [299]), which acts as a reference for global clock comparisons. Conversely, the idea for global gravity field determination using atomic clocks in space came up (e.g. Švehla/Rothacher [533]).

Bjerhammar [46], [47] can be viewed as one of the pioneers of relativistic geodesy. Unfortunately, previous occasional work on the same subject (Bjerhammar [45]) has been hardly appreciated by the geodesy community. Allan/Ashby [7] also already discussed a wide range of possible applications. Basic formulae for relativistic gravimetry and relativistic gradiometry have been derived by several authors (e.g. Soffel et al. [510], Kopejkin [295], Gill et al. [209], Kusche [304], Kopejkin et al. [299]).

Gravity measurements can be obtained by various means. If we apply classical absolute free-fall gravimeters, relativistic corrections have to be taken into account (Rothleitner/Francis [459]). By doing so, one tacitly retains Newton's interpretation of gravity. Stressing the smallness of relativistic effects, geodesists tend to treat general relativity as Newtonian mechanics plus relativistic corrections (Moritz/Hofmann-Wellenhof [369]). On the contrary, a consequent exploitation of Einstein's relativistic space-time concept almost inevitably leads to the idea of using clocks as a new gravity meter of choice, i.e., a tool for the purpose of height determination.

Realization of a GGOS implies the demand for a unified worldwide height system which relates to the geoidal geopotential value W_0, i.e., the geopotential of the equipotential surface which, in a least squares sense, fits best to the mean ocean level (Burša et al. [78], [79]). Thus W_0 can be defined by

$$\int_S (W - W_0)^2 \mathrm{d}S = \min, \tag{6}$$

where dS is a differential element of the oceanic surface topography.

In height computation W_0 is of paramount importance because it can be used to uniquely define the geoidal surface. Burša [77] provides a discussion on its relevance as a primary parameter in geodesy, and its current best estimate is given by Groten [220], [221]. Other important issues in the definition of a reference system of height are the metric unit (usually the SI meter), the exact relation between the terms 'height' and 'potential differences', and information about the tide model that has been in use. Height systems (either geometrically or physically defined) act as reference systems for the well-defined mathematical description of points in three-dimensional space in relation to two-dimensional height reference surfaces. The precise transformation between various height systems remains a central issue in contemporary physical geodesy.

Classical global height transfer is very much limited in accuracy due to the dynamic ocean topography (DOT) (Albertella et al. [5]). This is an hydrodynamical effect, basically non-gravitational, with magnitudes up to a few meters. Ideally, the sea level is assumed to be perpendicular to the plumb line direction. Due to DOT, this first order approximation is no longer adequate if we want to unify national height systems on an intercontinental or global scale. Another problematic issue is also related to $DOT = h - N$, where h is sea level height as determined by altimetry, and N denotes geoid height as being derived from gravity field determinations. Problems may arise, because h and N are of quite different nature. The former results from high-resolution point-wise direct altimetric measurements, whereas the latter is a derived quantity from a smoothed analytical function, i.e., worldwide gravity model.

In relativistic geodesy, we have to use a non-classical definition for the geoid. Bjerhammar [47] defines the relativistic geoid as the surface nearest to the mean sea level, on which precise clocks run with the same speed. Remark: one could apply alternative definitions of a relativistic geoid (Kopejkin [295], Kopejkin et al. [299]). For the use of GNSS leveling methods based on the concept of normal heights relating to the quasigeoid as a reference surface, precise computation of differences to orthometrical heights relating to the geoid as a reference surface is crucial (Wirth [597]). Flury/Rummel [191] exemplarily present necessary topographic mass computations for a precise geoid-quasigeoid separation. The quasigeoid plays an important role for the unification of regional height systems.

Determination of the global geoid is strongly connected to the issue of height levels or tide gauges and its location. Regarding the use of atomic clocks, future investigations have to reveal where to ideally place them on Earth and how to establish a link to already existing height systems. There seem to be different alternatives. One could locate them near the sea shores in order to ensure close connections to classical gauges. From an economical point of view, we may just leave the clocks at the laboratories of the metrological institutes for taking advantage of existing infrastructure, e.g., established frequency and time comparison instrumentation. Theoretically, it may be the best choice to set up new reference stations right in the middle of tectonically stable regions. By doing so, we could possibly reduce irregular disturbing effects on the clock readings. Other constraints may stem from time transfer issues like accessibility to glass fiber networks or potential satellite coverage. Using free air links requires considerations of atmospheric effects, etc.

Today, the geoid is determined either by space geodetic techniques, more precisely satellite gravity missions, or classical leveling, and their combination. On a global scale, consistency could be achieved on the decimeter level only, which is not accurate enough for a unified world height system. Ongoing gravity field space missions strive for the centimeter level and a spatial resolution of around 100 kilometers. The geoid is variable in time due to mass redistributions which are caused by tides (oceanic and solid), mass loading effects, hydrodynamics, plate tectonics, ice melting, etc. Amplitudes at the centimeter or decimeter level lead to uncontrollable frequency standard fluctuations in the order of 10^{-17} to 10^{-18}. These gravitational effects themselves do change time. Consequently, the question of the practical realization of a reference time and reference clock arises. If multiple reference clocks are very far apart, then potential changes due to variable solar influence (caused by Earth's rotation about its axis, thus leading to variations in the clocks' relative positions w.r.t. the Sun's gravitational potential) become significant and have to be accounted for. This would require extended time series of measurements and therefore long-stable atomic clocks.

26 The improvement of gravity field determination techniques

The determination of Earth's gravity field is a major issue in geodesy. Classical determinations comprise different sources of data which have to be combined (Fecher [178]). The merger of a satellite-derived geopotential models with gravity information available from the ground is discussed in Rapp/Pavlis [443]. For details on classical relations between reference surfaces, vertical datum, geoid, and spherical harmonic coefficients we refer to the literature (e.g. Rapp [444], Rummel/Teunissen [462]). A thorough introduction to potential theory can be found in Kellogg [272], and there exist specialized textbooks on potential theory in relation to geophysical applications (e.g. Blakely [50]). In the past, gravity and gravity gradient measurements were obtained solely either by earthbound classical gravimetry (e.g. Niebauer et al. [390]) or dedicated satellite gravity missions. The combination of gravity and altimetry mission results may also benefit from advanced atomic clock operations.

Proposals for future missions suggest novel metrology system detectors for gravity field determination, e.g., laser interferometer (Sheard et al. [502]) for a GRACE follow-on concept (Dehne et al. [127]). One goal of dedicated satellite missions is to accurately determine also the time rate of change of the spherical harmonic coefficients (Wahr [568]). Earth's gravity field is subject to change in time due to various reasons (Peters [414], [415], Kusche [305]). Growing demands in spatial and temporal resolution require innovative mission concepts for an improved gravity field determination. Several approaches, e.g., using optical techniques for both, SST and gradiometer test mass observations (Brieden et al. [68]), or alternative constellation scenarios and its implications (Elsaka/Kusche [171]) have been investigated in theory and by numerical simulations (Raimondo et al. [442]).

27 Further potential applications of highly precise atomic clocks

Atomic clocks may also be used for the comparison and the alignment of reference systems. The unification of different positioning and timing systems based on different satellite constellations would be supported. The same holds true for the unification of geometric (GNSS) and gravimetric (optical clocks) positioning. Precise positioning of spacecraft is of growing importance, e.g., in formation flying.

The timing aspect itself is another important issue. Accurate clocks in Earth orbit may provide a lasting and reliable global time scale which could be used, for instance, to monitor existing GNSS time signals. GNSS would be designed in a new way, based upon two-way connections. This allows for the elimination of the first order Doppler effects and enables real-time frequency dissemination. The annoying estimation of clock parameters and ambiguities would become unnecessary.

Precise timing is essential for Earth measurement techniques like reflectometry, radio occultations, scatterometry, atmospheric and ocean sounding. Popular research topics on climate change (e.g., separation of mass portions and volume portions in sea level changes), the atmosphere (e.g., atmospheric remote sensing by signal detection using zero differences), or oceanography (e.g. sea level heights, dynamic ocean topography, tsunami early warning systems) offer plenty of potential applications. The system as a whole is highly complex (e.g. Cazenave/Llovel [94]). Sea level changes result as a sum of many different causes, e.g., driven by tides, irregular Earth rotation, or meteorological fluctuations, and so on. Seasonal variations may contribute up to tens of centimeters. Over decades changes on the order of a few decimeters are possible. Eddies can spark off variations on the meter level. Additionally, large scale hydrodynamics (e.g. ocean currents) also changes the sea level on the order of one to two meters. Regarding climate research as a major GGOS application, atomic clocks may become a valuable tool for the separation of those geophysical effects.

Space geodetic techniques like Very Long Baseline Interferometry (VLBI) or Satellite Laser Ranging (SLR) are based on the precise determination of the signals' time of arrival, time delays, or travel times. Improved start/stop detection relies on better clocks. Using better clocks one could switch from relative to absolute epoch allocation and possibly decorrelate geometrically, atmospherically, and instrumentally induced time delays.

Furthermore, present time scale realizations at the ground stations are no better than on the $100\,ns$-level. So far, time scale synchronization between stations is done via GNSS. A closer tie of temporal and spatial referencing is highly desirable for existing and upcoming space geodetic techniques, e.g., a substantial gain in precision could make it feasible to perform VLBI also with solar system sources instead of only with quasars, or space based VLBI in addition to ground based interferometry. Time and frequency transfer in GNSS operations may even become precise enough to determine water vapour or higher order ionospheric effects.

Obviously, Earth system research in total with all its differential or time delay techniques would benefit from more accurate clock readings. Besides potential, i.e. height, measurements and positioning, the determination of the rotational behavior of a massive body (e.g. Earth) remains a major task in geodesy. Comparing the possible use of atom interferometers against customary laser technology shows that there is still lot of room for improvements regarding large ring lasers (Stedman et al. [520]). Besides space geodetic techniques, ring laser gyroscopes are currently being used to detect variations in Earth's rotation (Schreiber et al. [486]). Appropriate atomic clock networks could be another alternative device to sense rotation.

A general advantage of the chronometric approach is the fact, that we do not necessarily rely on available line of sights. This may be of great benefit if we think of underwater or tunnel applications.

In the long term, the sensitivity of clocks will probably reach a level where very moderate velocities become detectable using the relativistic/gravitational Doppler effect. In this case a lot more applications (e.g. monitoring of plate tectonics) come in reach.

28 The relativistic approach in satellite orbit calculation

In addition to the above mentioned examples of Earth measurements, relativistic effects are routinely to be applied in satellite geodesy (Müller et al. 2008), and especially in satellite orbit calculations (Hugentobler [253]). Combrinck [106] provides an overview of (general) relativistic effects that are essential for selected space geodetic techniques, e.g., for GNSS like GPS (Ashby [21]) or gravity field space missions like GRACE (Larson et al. [322]).

The relativistic equations of motion of an artificial Earth orbiting satellite, e.g. the relativistic Schwarzschild problem, can be solved by various methods comprising different sets of orbital elements. In practice, dealing with real orbital data most likely leads to several metrics that have to be involved in the formulation of the whole motion problem. In this respect, Georgevic/Anderson [207] provide advantageous relationships between osculating elements for the general relativistic Schwarzschild problem. The relativistic N-body problem was first treated in the late 1930's (e.g. Einstein et al. [164], Eddington/Clark [152]).

In general, there are different ways to solve the relativistic two body problem (Kopejkin et al. [299]), e.g., by Fourier series expansion (Broucke [69]). For approaches that require the Hamiltonian of the system in post-Newtonian approximation, Schäfer [468] provides the 2PN-Hamiltonian, whereas the 3PN-Hamiltonian is given in Damour et al. [119]. Details on the solution in context of the PPN theory are given in Soffel et al. [508].

So far, applications of the relativistic two body problem are mainly discussed in astronomy, e.g., for the study of compact binary star systems. The Lagrangian up to third post-Newtonian order (Blanchet [52]) can be used to push the approximation level to a higher order. One needs to have a very precise two body solution at hand before observational data can be surveyed for very week relativistic effects like the predicted gravitational waves (Blanchet [53]). By increasing the degree of freedom, Barker/O'Connell [31] discuss various precession effects within the two body problem from a Lagrangian point of view in consideration of a possible spin of both bodies and an additional quadrupole moment of the primary body.

Deep space navigation of interplanetary spacecraft by means of precise clock readings in a changing gravitational potential is just another conceivable space geodetic application.

Present space missions comprising highly precise atomic clocks mostly still focus on relativity tests. The ACES mission (Švehla [534]) shall demonstrate the potential of relativistic geodesy by performing ground-to-space clock comparisons. Two frequency standards will be attached to the ISS, a laser cooled cesium atomic clock and a hydrogen maser. The time transfer is based on a stable microwave link. Another task of ACES is to provide a precise orbit determination for the ISS. Based on ACES technology, upgraded frequency standards will be used in the Space-Time Explorer and Quantum Equivalence Principle Space Test (STE-QUEST) mission (Cacciapuoti [81]). The earthbound orbit will be highly elliptical in order to gain differences in potential and velocity values along the trajectory as large as possible. This leads to easier testable relativistic effects. Remark: the principle idea for the determination of the gravitational redshift by means of an Earth orbiting satellite already came up in the early 1970's (Kleppner et al. [283]).

Outline of the mathematical framework

Einstein stressed the equivalence of all reference systems. Physical laws should be expressed by equations that describe geometrical entities and relations independent of the reference system. They should be invariant with respect to coordinate transformations and with respect to the curvature of space-time. Physical laws must reflect the invariant properties of space-time geometry. This can be achieved by the use of tensor relations.

As we have learned from Einstein and others, there exist very strong ties between (differential) geometry and relativity. The index notation is an indispensable feature in the theory of relativity. Therefore, it is no surprise that tensor calculus is also of advantage in differential geometry (Eisenhart [167]) or linear algebra. One of the main advantages is the fact that resulting formulas will hold for any number of dimensions. The well-established theory of surfaces can readily be extended to space or even space-time, and possibly beyond, e.g. for an extended phase space. Applications comprise ellipsoidal geometry, differential geometry of the gravity field, the theory of refraction, or the connection of relativistic and quantum mechanical aspects. One of the first attempts to fully exploit the potential of this new approach to geodesy was done by Marussi [348].

In order to employ the general relativistic approach, one has to extend the classical concepts of differential geometry. Geodesists are very familiar with the generalization of flat Euclidean geometry to the curved Riemannian geometry of (two-dimensional) surfaces with its extrinsic notion of curvature, as well as with Newtonian physics. However, its restrictions to invariant time intervals and invariant space intervals (implying the existence of an absolute simultaneity) are not compatible with the precision level of atomic clock observations. Instead, we have to apply the (special) relativistic concept of invariant space-time intervals $\mathrm{d}s$, the study of which enables the derivation of the space-time's structure. Special relativity is understandable as a theory of flat Lorentzian geometry, where we still fix the geometry of space-time in advance. Likewise to classical differential geometry, we can generalize to a curved Lorentzian geometry, where the geometry of space-time dynamically evolves. This, in combination with the equivalence principle, is the fundamental idea of general relativity. It finally leads to the dynamical evolution equation for the metric, i.e., Einstein's equation. In comparison to classical differential geometry, one generalizes the Riemannian geometry to non-positive definite metrics, which leads to a purely intrinsic notion of curvature and the existence of null-vectors (non-zero vectors with zero-„length").

This chapter is not intended to substitute for the reading of detailed textbooks on relativity and/or differential geometry. Following Wald [569] a precise mathematical introduction to the mathematical framework of relativity requires a somewhat laborious step-wise derivation of several notions and definitions. In this paragraph we shortly recapitulate the main steps that will be outlined in more detail within the preceding sections.

One usually starts with the precise notion of a set of points that constitutes space-time by the notion of a manifold, still without any metrical or other structure. Then, the notion of a tangent vector has to be introduced, which can be done in several (more or less intuitive and direct) ways, e.g., via a directional derivative operator (acting on functions). The following step, the definition of tensors of arbitrary rank, in general relativity differs from the usual procedure that relies on the existence of a positive definite inner product and the expression of tensor components in an orthonormal basis. Instead, within the theory of relativity it is essential to make no assumption on the inner product in advance, i.e., the role of the space-time metric is completely explicit, it acts as the key unknown variable to be solved for. Next one defines a dual vector space from a given finite-dimensional vector space, the latter being a tangent space at a point of space-time. In general relativity, space-time does not have the structure of a vector space, and coordinates are merely labels of events in that space-time. Now one should introduce a tensor over an arbitrary vector space, and study basic operations that can be performed on tensors (contraction, taking outer products). A metric on a vector space is defined as a special type of tensors. If it is positive-definite, it describes ordinary curved geometries (Riemannian metric). In mixed cases (e.g., negative-definite on a one-dimensional subspace and positive-definite on the orthogonal complement of this subspace), it describes general relativity's curved space-time (Lorentzian metric) via the inner product of tangent vectors. The space-time metric, at a later stage, will be used for the determination of elapsed proper time along a time-like curve.

Different tensor notations do exist, here we will mostly follow the traditional index notation, acknowledging that this approach to a certain extent veils the true nature of a tensor but allows for easier handling in practice. Furthermore, one has to define a tensor field and introduce a notion of its differentiation. This is a subtle task, because we could define the derivative in various ways, e.g., via a notion of parallel transport of vectors along a curve. Consequently, one can then define a geodesic as a curve whose tangent is parallel transported. One adds some structure to the tensor field by imposing an additional requirement in stipulating that the derivative of the metric shall be zero. This step introduces the concept of a co-variant derivative. Analogically with Riemannian geometry (where geodesics can be considered as curves of extreme length with respect to variations whilst keeping its endpoints fixed), in Lorentzian geometry time-like geodesics can be characterized as a curve that extremizes the elapsed proper time along the curve.

Finally, curvature can be defined in different but equivalent ways. Curvature is (Wald [569])

- the failure of successive co-variant derivatives on tensor fields to commute,
- the failure of parallel transport of a vector around a closed curve to return the vector to its original value,
- the failure of initially parallel and nearby geodesics to remain parallel.

We will mathematically describe curvature via the Riemann curvature tensor. Aiming at a geodetic exploitation of clock readings and its comparisons (via signal transmission or clock transport), it is essential to determine time-like and null geodesics in space-time, respectively. The former represent possible paths of freely falling particles (to study the motion of observers, clocks, celestial bodies, etc.), whereas the latter represent possible paths of light rays (to study the travel of signals, bending of light, etc.). The determination process basically involves the solution of the respective geodesic equations.

Remark: again, an essential feature of general relativity is the non-existence of any non-dynamical background structure of space-time. Therefore one should theoretically use a completely coordinate independent way for its formulation. On the other hand, this text first and foremost aims at a geodetic readership. Herein we will intentionally get (slightly) off the before-mentioned strict formal mathematical path. Instead, we will introduce coordinates at the outset and mostly work with components of tensors in coordinate bases, define differentiation of tensors via the introduction of the Christoffel symbol, and use these to set up the Riemann curvature tensor.

The first half of this chapter introduces the basic relativistic framework along the lines of differential geometry concepts, following a few standard texts, e.g. Moritz/Hofmann-Wellenhof [369]. The second half accentuates those parts that are of importance for relativistic geodesy, especially various formulations of the gravity potential. A subsequent chapter exemplarily focusses on the chronometric leveling idea.

29 Introduction of fundamental relations

29.1 Equation of a geodesic

One and the same geometric vector may look algebraically different, depending on the chosen coordinate system. In opposition to algebraic vectors (i.e. n-tuples of numbers), geometric vectors (i.e. arrows in n-dimensional space) are independent of the coordinate system. Index notation as a powerful alternative to symbolic vector or matrix notation becomes especially useful, if affine coordinates are involved. Base vectors $\mathbf{b}_i$ are no longer unit vectors and/or are not all mutually orthogonal, i.e. $\mathbf{b}_i \cdot \mathbf{b}_j \neq \delta_{ij}$.

In this case we may describe a certain vector $\mathbf{x}$ either by covariant components x_i or contravariant components x^i. The former are obtained by orthogonal projection of the geometric vector onto the base vectors, whereas the latter are obtained by parallel projection. Introducing Einstein's summation convention as well as covariant ($\mathbf{b}_i$) and contravariant ($\mathbf{b}^i$) base vectors, we get specifically for the three-dimensional space

$$\mathbf{b}^1 = \frac{\mathbf{b}_2 \times \mathbf{b}_3}{(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)}, \quad \mathbf{b}^2 = \frac{\mathbf{b}_3 \times \mathbf{b}_1}{(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)}, \quad \mathbf{b}^3 = \frac{\mathbf{b}_1 \times \mathbf{b}_2}{(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)} \tag{7}$$

with

$$\mathbf{b}_i \cdot \mathbf{b}^j = \mathbf{b}^i \cdot \mathbf{b}_j =: g_i^j = \delta_i^j = \delta_{ij} = \delta^{ij}, \tag{8}$$

where $[g_i^j] = [\delta_i^j]$ (fundamental mixed tensor) is identical to the identity matrix. The vector $\mathbf{x}$ can be written as

$$\mathbf{x} = x^i \mathbf{b}_i = x_i \mathbf{b}^i. \tag{9}$$

The distinction between covariant and contravariant indices is not needed, if only rectilinear coordinates are being used. The main advantage of tensors however is the fact, that they represent geometrically invariant objects. Again, all formulas will equally work for any n-dimensional space and for any choice of curvilinear coordinates. Index notation is superior in geometrical interpretation, whereas in many cases, symbolic notation is easier to use in calculations. Almost any textbook on relativity and gravitation (e.g. Ohanian [395]) comprises at least some kind of introduction to tensor calculus for non-orthonormal coordinate systems (i.e. Ricci calculus). Here we only highlight those relationships that affect our purpose of calculations in relativistic geodesy.

To start with, we mention the fundamental covariant and contravariant metric tensor, given by

$$g_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j, \quad g^{ij} = \mathbf{b}^i \cdot \mathbf{b}^j \quad \Rightarrow \quad \left[g^{ij}\right] = \left[g_{ij}\right]^{-1}. \tag{10}$$

They can be used to switch between covariant and contravariant vectors by lowering or raising of indices (remember Einstein's summation convention):

$$\mathbf{b}_i = g_{ij}\mathbf{b}^j, \quad \mathbf{b}^i = g^{ij}\mathbf{b}_j, \qquad x_i = g_{ij}x^j, \quad x^i = g^{ij}x_j. \tag{11}$$

In special relativity we have constant metric tensors which is equivalent to a linear space. Non-linearity would refer to the coordinate system (curvilinear coordinates) and/or to the space itself (curved space). In ordinary differential geometry (position vector of a surface $\mathbf{x} = \mathbf{x}(u^1, u^2)$ with partials $\mathbf{x}_{u^i} := \mathrm{d}\mathbf{x}/\mathrm{d}u^i$), the (covariant) metric tensor is directly related to the Gaussian fundamental quantities of first kind E, F, G via (Heck [237])

$$[g_{ij}] = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad \text{with} \quad \begin{array}{l} E(u^1,u^2) := \mathbf{x}_{u^1}\cdot\mathbf{x}_{u^1}, \\ F(u^1,u^2) := \mathbf{x}_{u^1}\cdot\mathbf{x}_{u^2}, \\ G(u^1,u^2) := \mathbf{x}_{u^2}\cdot\mathbf{x}_{u^2}, \end{array} \quad \text{and} \quad \mathrm{d}s^2 = E(\mathrm{d}u^1)^2 + 2F\,\mathrm{d}u^1\,\mathrm{d}u^2 + G(\mathrm{d}u^2)^2. \tag{12}$$

Tensor algebra deals with the calculus of constant metric tensors in Euclidean spaces using linear coordinates (cartesian or affine), whereas tensor analysis in applied for the calculus of variable tensors in curvilinear coordinates (Moritz/Hofmann-Wellenhof [369]).

In geodesy, the concept of a geodesic as the straightest possible connecting curve between two points is essential (e.g. Mai [338]). The fundamental parameter arc length s of a curve is chosen as the independent variable. Alternatively, this metric quantity can be related to other (physical) quantities, e.g. time.

In rectangular coordinates, the differential equation of a straight line is given by

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2} = \mathbf{0}, \tag{13}$$

because the solution is simply $\mathbf{x} = \mathbf{c}_1 s + \mathbf{c}_0$, where $\mathbf{c}_0$ and $\mathbf{c}_1$ are the constants of integration. As in the case of ordinary differential geometry, the position vector $\mathbf{x}$ of a (surface) curve is given by some (surface) parameters u^i. In general, these parameters may represent curvilinear coordinates. By making use of the tensor calculus, we are no longer restricted to the treatment of surfaces ($i = 1, 2$) but can deal with spaces ($i = 1, 2, 3$), space-times ($i = 0, 1, 2, 3$) or any n-dimensional space in general. The generalized final form of a straight line (geodesic) in curvilinear coordinates results in

$$\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + \Gamma^i_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}s}\frac{\mathrm{d}u^k}{\mathrm{d}s} = 0, \tag{14}$$

where the indices i, j and k all have the same range. For simplicity reasons one introduces so-called Christoffel symbols of first

$$\Gamma_{kij} := \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \tag{15}$$

and second

$$\Gamma^l_{ij} := g^{lk}\Gamma_{kij} \tag{16}$$

kind. Instead of using Γ, several other notations exist. Due to the commutativity of the inner product of the base vectors, the metric tensors and hence the Christoffel symbols show some properties of symmetry. Therefore, in practice, we do not have to explicitly calculate its values for all possible n^3 index combinations.

The generalized square of the line element takes the form (with $g_{ij} = g_{ij}(u^i)$)

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}u^i\mathrm{d}u^j. \tag{17}$$

Equation (14) can be regarded as a special case of

$$\frac{\mathrm{d}^nu^i}{\mathrm{d}s^n} + \Gamma^i_{jkl\cdots}\underbrace{\frac{\mathrm{d}u^j}{\mathrm{d}s}\frac{\mathrm{d}u^k}{\mathrm{d}s}\frac{\mathrm{d}u^l}{\mathrm{d}s}\cdots}_{\text{n factors}} = 0\,, \tag{18}$$

where the unit tangent vectors $\mathrm{d}u^i/\mathrm{d}s =: \xi^i$ could be used to define Riemannian coordinates via $x^i := \xi^i s$, acting as cartesian coordinates in a tangent plane.

Higher Christoffel symbols are defined as

$$\Gamma^i_{jkl} := \frac{\partial\Gamma^i_{jk}}{\partial u^l}\,, \quad \Gamma^i_{jklm} := \frac{\partial\Gamma^i_{jkl}}{\partial u^m}\,, \quad \text{etc.} \tag{19}$$

and the general solution can be expressed as a series expansion

$$u^i = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{d}^nu^i}{\mathrm{d}s^n}\right)\bigg|_0 s^n = u^i\Big|_0 + \xi^i\Big|_0 s - \frac{1}{2}\Gamma^i_{jk}\xi^j\xi^k\Big|_0 s^2 - \frac{1}{6}\Gamma^i_{jkl}\xi^j\xi^k\xi^l\Big|_0 s^3 - \frac{1}{24}\Gamma^i_{jklm}\xi^j\xi^k\xi^l\xi^m\Big|_0 s^4\cdots. \tag{20}$$

In mathematical geodesy, the explicit differential equations for a geodesic in case of $n = 2$ (geodesics as surface curves) and orthogonal coordinates are very well-known. They are related to so-called main geodetic problems, namely the direct (initial value problem) and inverse, i.e. indirect (boundary value problem) (e.g. Heck [237]). There exist several techniques to solve these, e.g., via a Hamiltonian approach (Mai [338]). For our purposes equation (14) is of great importance, because it also represents the trajectory of a point in the gravitationally curved space-time of general relativity.

Another way to look at equation (14) is the idea of parallel transport of a contravariant vector v^i, i.e. $\mathbf{v} = v^i \mathbf{b}_i$, which means that its direction and length do not change during transport. Ordinary differentiation of $\mathbf{v}$ yields

$$\mathrm{d}\mathbf{v} = \big(\mathrm{d}v^i + \Gamma^i_{jk} v^j \mathrm{d}u^k\big)\mathbf{b}_i =: \mathrm{D}v^i \mathbf{b}_i \tag{21}$$

and therefore, in case of a parallel transport,

$$\mathrm{d}\mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathrm{d}v^i + \Gamma^i_{jk} v^j \mathrm{d}u^k = 0. \tag{22}$$

Dividing the whole equation by $\mathrm{d}s$ and regarding as a special case the tangent vector of the curve, namely $v^i = \mathrm{d}u^i/\mathrm{d}s$, we end up with (14) again. Thus, along a geodesic (generalized straight line), the tangent vector of the curve has constant direction. In other words, geodesics are autoparallel.

The intrinsic differential of a contravariant vector

$$\mathrm{D}v^i = \left(\frac{\partial v^i}{\partial u^k} + \Gamma^i_{jk} v^j\right)\mathrm{d}u^k \tag{23}$$

can be used to further shorten the general differential equation of a geodesic. Using the intrinsic derivative $\mathrm{D}/\mathrm{D}s$ in combination with the ordinary derivative $\mathrm{d}/\mathrm{d}s$ one gets equivalently to equation (14) (Stephani [523])

$$\frac{\mathrm{D}}{\mathrm{D}s}\frac{\mathrm{d}u^i}{\mathrm{d}s} = 0. \tag{24}$$

29.2 Riemannian curvature tensor

The first fundamental form (17) in differential geometry reflects the intrinsic geometry of a surface. It is governed by the metric tensor g_{ij}. To describe its curvature in the embedding space we apply the second fundamental form by making use of the Gaussian fundamental quantities of second kind, forming the tensor L_{ij} via

$$\big[L_{ij}\big] = \begin{bmatrix} L & M \\ M & N \end{bmatrix}, \tag{25}$$

where L_{ij} can be calculated using the surface normal vector $\mathbf{z}$ by

$$L_{ij} = \frac{\partial^2 \mathbf{x}}{\partial u^i \partial u^j} \cdot \mathbf{z}. \tag{26}$$

All important theorems of differential geometry can be generalized using the index notation. As an example, the curvature of a normal section $\kappa^n = 1/R$ with R being the radius of curvature, as usually given by the theorem of Euler, now reads

$$\frac{1}{R} = L_{ij}\frac{\mathrm{d}u^i}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s} = \frac{L_{ij}\,\mathrm{d}u^i \mathrm{d}u^j}{g_{ij}\,\mathrm{d}u^i \mathrm{d}u^j}\,. \tag{27}$$

The parallel transport in space, as defined by equation (22), has to be supplemented with a subsequent normal projection onto the tangent plane if a (curved) surface is involved. As a consequence, the parallel transport of a surface vector becomes path dependent and is directly related to the curvature of the surface. Generalizing this result to higher dimensions, we are thus able to set up experiments where we can sense and quantify the curvature of space-time by geodetic measurements.

The condition for parallel transport based upon the intrinsic differential can be stated for both, the contravariant and covariant representation of a vector:

$$\begin{aligned} \mathrm{D}v^i &= \mathrm{d}v^i + \Gamma^i_{jk} v^j \mathrm{d}u^k = 0, \\ \mathrm{D}v_j &= \mathrm{d}v_j - \Gamma^i_{jk} v_i \mathrm{d}u^k = 0. \end{aligned} \tag{28}$$

The magnitude of the deviations caused by path dependent parallel transportation of a (surface) vector is a direct measure of the (surface) curvature and can be expressed by the Riemannian curvature tensor. Its contravariant form is defined as

$$R^i_{jkl} := \frac{\partial \Gamma^i_{jl}}{\partial u^k} - \frac{\partial \Gamma^i_{jk}}{\partial u^l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}\,, \tag{29}$$

where all indices have the same range from 1 to n (overall dimension). If time is involved, the range may also start from 0. Regarding 4-dimensional space-time, traditionally we use $i, j, k, l, m = 0, 1, 2, 3$. The covariant form follows from

$$R_{ijkl} = g_{im} R^m_{jkl} \,. \tag{30}$$

If all components of the Riemannian curvature tensor are zero, we face a totally flat n-dimensional space, otherwise it may be curved to some extent. Depending on the actual non-zero components, the curvature can also be related to a sub-space only. Having four indices, this tensor consists of n^4 components (total number of index combinations). But, due to symmetry reasons, only $n^2(n^2-1)/12$ of them are potentially non-zero and independent of each other. So, in case of a 2-dimensional surface, a 3-dimensional space, or a 4-dimensional space-time, we just need to calculate 1, 6, or 20 different components, respectively, to judge about its intrinsic geometrical properties. In compliance with ordinary differential geometry, we get a single intrinsic measure for the curvature of a surface. The well-known Gaussian curvature K is directly related to the Riemannian curvature tensor and the determinant of the metric tensor $|g_{ij}|$ via

$$K = \frac{R_{1212}}{|g_{ij}|} \,, \tag{31}$$

and the famous theorema egregium of Gauss can formally be derived by the index notation approach. The Riemannian curvature may be regarded as a generalization of the Gaussian curvature to higher dimensions. As a remark, sometimes the determination of the metric from a given curvature is called the inverse problem (Quevedo [439]). The same paper also discusses different ideas to measure the curvature of space-time.

Geodesics are fundamental not only in Gauss' theory of surfaces or within Riemannian geometry but also in Einstein's theory of relativity. As Moritz/Hofmann-Wellenhof [369] remark, depending on an intrinsic or extrinsic point of view, they represent autoparallel curves, shortest lines, straightest lines, and the force free motion on a (n-dimensional) surface, or they can be related to a higher-dimensional embedding space. In the theory of surfaces, the shortest possible definition of a geodesic is given by

$$\mathbf{n} = \pm\,\mathbf{z} \,, \tag{32}$$

where $\mathbf{n}$ is the principal normal of the curve, $\mathbf{z}$ still denotes the surface normal. Equation (32) directly leads to the defining ordinary differential equation(s) of a geodesic (either second order or multiple first order equations). These are especially suited for the solution of the direct main geodetic problem (Heck [237]). Alternatively, a geodesic may also be characterized by a partial differential equation for the distance, namely the eiconal equation (Eisenhart [167]), which fits very well to the treatment of the inverse main geodetic problem (Heck [237]):

$$g^{ij} \frac{\partial s}{\partial u^i} \frac{\partial s}{\partial u^j} = 1. \tag{33}$$

29.3 Eötvös tensor and Marussi tensor

In physical geodesy, the occurrence of non-linear three-dimensional coordinates is quite common. As spatial parameters we could choose astronomical latitude Φ and longitude Λ, and Earth's gravity potential value W as kind of natural coordinates, such that

$$u^1 = \Phi, \quad u^2 = \Lambda, \quad u^3 = W. \tag{34}$$

Relativistic geodesy is strongly related to the differential geometry of gravity fields. The second partial derivatives of the gravity potential form a (symmetric) gravity gradient tensor, known as Eötvös tensor

$$E := \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} = \begin{bmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{bmatrix} \quad \text{with} \quad W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j} \,, \tag{35}$$

whose elements can be measured by different means, e.g. gradiometry. As a remark, Torge [550] uses a slightly different sign convention, where $\mathbf{g} = \nabla W = (W_x, W_y, -W_z)^T$ such that the Eötvös tensor results as

$$\nabla \mathbf{g} = \begin{bmatrix} W_{xx} & W_{xy} & -W_{xz} \\ W_{yx} & W_{yy} & -W_{yz} \\ -W_{zx} & -W_{zy} & -W_{zz} \end{bmatrix}. \tag{36}$$

The cartesian coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$ form an orthogonal coordinate system originating at the (surface) point P of interest as defined by Φ_P, Λ_P, W_P, and axes pointing towards north, east, and vertical upwards.

Eötvös' tensor W_{ij} bears physical dimension $1/T^2$. In order to relate gravity with geometry, one introduces the Marussi tensor w_{ij}, which bears geometrical dimension $1/L$, by $[w_{ij}] = -[W_{ij}]/g$ (Marussi [348]), where g represents the acceleration due to gravity. The individual components of the Marussi tensor can now be interpreted in terms of curvature and torsion. Introducing orthometric height H^o, $\partial/\partial z \equiv \partial/\partial H^o$ and $g = -W_z$ one gets

$$[w_{ij}] = \frac{1}{g}\begin{bmatrix} -W_{xx} & -W_{xy} & \partial g/\partial x \\ -W_{yx} & -W_{yy} & \partial g/\partial y \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} \kappa_x^n & \tau_x & \kappa_x \\ \tau_x & \kappa_y^n & \kappa_y \\ \kappa_x & \kappa_y & w_{33} \end{bmatrix}. \tag{37}$$

The quantities κ_x^n, κ_y^n (normal curvature in meridional and parallel direction) and τ_x (geodesic torsion in meridional direction) indicate the curvature of the level surface at P. Furthermore, κ_x and κ_y represent the curvature of the plumb line through P. Taking the vertical gradients of our natural coordinates, we find

$$\begin{aligned} w_{13} = \kappa_x &= \frac{1}{g}\frac{\partial g}{\partial x} = \frac{\partial \Phi}{\partial H^o}, \\ w_{23} = \kappa_y &= \frac{1}{g}\frac{\partial g}{\partial y} = \cos\Phi \frac{\partial \Lambda}{\partial H^o}, \\ w_{33} &= \frac{1}{g}\frac{\partial g}{\partial z} = \frac{1}{g}\frac{\partial g}{\partial H^o}. \end{aligned} \tag{38}$$

The mean curvature of the level surface can be calculated as $H = (\kappa_x^n + \kappa_y^n)/2$. The tensor L_{ij} allows us to express mean curvature $H = \mathrm{tr}[L_{ij}]/2$ and Gaussian curvature $K = \det[L_{ij}]$ directly. Constituting the upper left part of the Marussi tensor, L_{ij} also relates to the so-called Dupin indicatrix (Moritz/Hofmann-Wellenhof [369]) and connects genuine geodetic concepts (e.g. direction of north) with pure surface theory concepts via Pizzetti's theorem. Denoting the angle from the northern direction towards the meridional curve by m, and towards the parallel curve by p, respectively, we get

$$\tan m = -\frac{w_{12}}{w_{22}}, \quad \tan p = -\frac{w_{11}}{w_{12}}. \tag{39}$$

Taking the gradient of g with respect to the local cartesian coordinate system, and performing a subsequent comparison of coefficients will eventually lead to the most important formula of intrinsic geodesy for practical applications, namely Bruns' equation. It relates the geodetically important vertical gradient of gravity $\partial g/\partial H^o$ to genuine geometrical (mean curvature H), physical (gravitational constant G, mass density ρ), and astronomical quantities (rotational velocity of Earth $\omega_\oplus$):

$$\frac{\partial g}{\partial H^o} = -2gH + 4\pi G\rho - 2\omega_\oplus^2. \tag{40}$$

In classical geodesy, (orthometric) heights are determined indirectly by measurements of changes in gravity, i.e. potential differences ($\delta W = -g\,\delta H^o$). The gravity vector itself results as the gradient of the gravity potential, i.e. $\mathbf{g} = \mathrm{grad}\,W$. Relativistic geodesy enables us to determine (orthometric) heights directly by measuring the local properties of space-time using highly precise atomic clocks. The Marussi tensor is strongly related to the metric tensor(s). Inversely, we can express the metric tensor(s) in terms of the w_{ij}. In (the easier) case of the contravariant metric tensor one finds

$$[g^{ij}] = \begin{bmatrix} w_{11}^2 + w_{12}^2 + w_{13}^2 & \dfrac{(w_{11}+w_{22})w_{12} + w_{13}w_{23}}{\cos\Phi} & -g\,w_{13} \\ \dfrac{(w_{11}+w_{22})w_{12} + w_{13}w_{23}}{\cos\Phi} & \dfrac{w_{12}^2 + w_{22}^2 + w_{23}^2}{\cos^2\Phi} & -\dfrac{g\,w_{23}}{\cos\Phi} \\ -g\,w_{13} & -\dfrac{g\,w_{23}}{\cos\Phi} & g^2 \end{bmatrix}. \tag{41}$$

For practical computations, one could set up an auxiliary matrix

$$M := \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ 0 & 0 & 1 \end{bmatrix}, \tag{42}$$

perform the matrix multiplication

$$MM^T = \begin{bmatrix} w_{11}^2 + w_{12}^2 + w_{13}^2 & w_{11}w_{21} + w_{12}w_{22} + w_{13}w_{23} & w_{13} \\ w_{11}w_{21} + w_{12}w_{22} + w_{13}w_{23} & w_{21}^2 + w_{22}^2 + w_{23}^2 & w_{23} \\ w_{13} & w_{23} & 1 \end{bmatrix}, \tag{43}$$

and alter the resulting matrix by the following rules. Multiply every element

- with index 2 occurring k times by the factor $(1/\cos\Phi)^k$,
- with index 3 occurring k times by the factor $(-g)^k$.

29.4 Ricci curvature tensor and fundamental metric tensors

The remarks following (30) imply that we may compress the information content of the Riemannian curvature tensor to some extent. It is common practice to contract the Riemannian curvature tensor (rank 4) into the Ricci (curvature) tensor (rank 2) via

$$R_{jk} = g^{il} R_{ijkl}. \tag{44}$$

As before, this new curvature tensor is symmetric. Out of its n^2 components (total number of index combinations) only $n(n+1)/2$ have to be calculated. For a curved space $R_{jk} \neq 0$ has to be true for at least one index combination, otherwise it is a flat space.

We have already noticed, that only for $n = 2$ a single quantity (Gaussian curvature K) is sufficient to characterize the (in general, position dependent) curvature. For higher dimensions curvature will become direction dependent, too. An n-dimensional surface element can be regarded as being part of a generalized tangent plane, which is spanned by two generalized vectors ξ^i and η^i. Their directions depend on the direction of the generalized surface normal z^i. As a consequence, we can generalize Gaussian curvature to be applicable to any dimension n and call it Riemannian curvature. This direction dependent quantity (likewise denoted by K)

$$K = \frac{R_{ijkl}\xi^i\eta^j\xi^k\eta^l}{(g_{ik}g_{jl} - g_{il}g_{jk})\xi^i\eta^j\xi^k\eta^l} \tag{45}$$

is identical to the Gaussian curvature in case of $n = 2$, and thus $R_{ij} = -Kg_{ij}$. In three-dimensional space equation (45) simplifies to $K = R_{ij}z^iz^j$. In any case we recognize a close relation between the Ricci curvature tensor and the fundamental metric tensor(s).

29.5 Line element and special relativity

In special relativity the metric tensor reduces to a constant diagonal matrix $[\eta_{ij}] = \mathrm{diag}[-1, 1, 1, 1]$, such that

$$\mathrm{d}s^2 = \eta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2. \tag{46}$$

To avoid an unnecessarily messy notation, we temporarily switch to lower indices when writing formulas explicitly. The actual definition of the fundamental tensor η_{ij} is not fixed. There exist other forms of the metric that are equally valid. For the illustration of the line elements' invariance with respect to a Lorentz transformation one usually introduces

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2. \tag{47}$$

With settings $x = x_1$, $y = x_2$, $z = x_3$, $ct = x_0$ equation (46) follows immediately. This line element does not describe a pure Euclidean metric but rather a pseudo-Euclidean metric, because a Lorentz transformation leaves a hyperbola invariant instead of a circle. This has to be kept in mind when interpreting properties of world lines within Minkowski diagrams. From a theoretical point of view, it may be of advantage to introduce a Euclidean line element in a four-dimensional Minkowski space. This can be achieved by substituting $x_4 = ix_0 = ict$ such that $\mathrm{d}x_4^2 = -\mathrm{d}x_0^2$, resulting in the Minkowski form

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2. \tag{48}$$

Thus we can apply the Pythagorean theorem in four-dimensional space, but only at the price of dealing with a non-physical imaginary time. Nonetheless, depending on the problem at hand, one may easily switch from one interpretation or form to another.

Due to the variable sign of $\mathrm{d}s^2$, three different kinds of a world line (of an hypothetical observer), i.e., geodesics in four-dimensional space time, do exist. Differences occur depending on the observers' velocity. Light propagates along geodesics where $\mathrm{d}s^2 = 0$, therefore we name it *light-like* geodesics. On the contrary, in the state of rest in time, we have $\mathrm{d}t = 0$ and the line element reduces to an ordinary spatial line element $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 > 0$. Representing a straight line in $\mathbb{R}^3$, we call it a *space-like* geodesic. In special relativity, the straight world line of an observer at rest with respect to some inertial system represents a *time-like* geodesic with $\mathrm{d}s^2 < 0$. In order to avoid an imaginary quantity, a purely formal factor i (imaginary unit) is being introduced together with the observers' real-valued proper time τ via

$$\mathrm{d}s =: ic\mathrm{d}\tau \tag{49}$$

such that $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$.

29.6 Proper time and generalized Doppler effect

Denoting the spatial velocity of the observer as $v = \mathrm{d}r/\mathrm{d}t$ we find a fundamental relationship between coordinate time t and proper time τ, as given by an atomic clock:

$$-c^2\mathrm{d}\tau^2 = \left(v^2 - c^2\right)\mathrm{d}t^2 \quad \Rightarrow \quad \mathrm{d}\tau = \sqrt{1 - \left(\frac{v}{c}\right)^2}\,\mathrm{d}t. \tag{50}$$

This equation indicates that a time dilatation will occur whenever an object, e.g. an atomic clock, is moving with respect to the t-related coordinate system, i.e., the rest frame $(x, y, z; t)$. Because of $v < c$, always $\mathrm{d}\tau < \mathrm{d}t$.

Time dilatation can be regarded as a relativistic transversal Doppler effect (Moritz/Hofmann-Wellenhof [369]). Even in the absence of the classical Doppler effect at the point of closest approach, emitted signals will undergo a frequency shift simply due to the transmitters' relative velocity with respect to a receiving observer at rest. Integrating equation (50), the emitted frequency $f_E = 1/\Delta\tau$ and received frequency $f_R = 1/\Delta t$ are related by

$$f_R = \sqrt{1 - \left(\frac{v}{c}\right)^2}\, f_E \quad \Rightarrow \quad f_R < f_E\,. \tag{51}$$

There is another relativistic phenomenon that has to be taken into account, namely a longitudinal Doppler effect. It also includes the classical Doppler effect. The physical reason for the longitudinal effect is the limited travel speed of the signal. It takes some travel time (travel distance r divided by travel speed c) to reach the observer. For a moving source, the distance becomes time and velocity dependent, i.e. $r = vt$, such that the reception time will be $t_R = t + r/c = t + vt/c = (1 + v/c)t$. This relation is equally valid for some corresponding time intervals Δt_R and Δt. The latter can now be replaced by $\Delta\tau$, leading to

$$\Delta t_R = \left(1 + \frac{v}{c}\right)\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\,\Delta\tau = \frac{c + v}{\sqrt{c^2 - v^2}}\,\Delta\tau. \tag{52}$$

Replacing time intervals by its respective frequencies yields

$$f_R = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\, f_E\,. \tag{53}$$

In order to unveil the classical Doppler effect, one may replace the square root by a product of binomial series expansions (for the nominator and denominator) and neglect terms of higher order, i.e., starting with $O\big((v/c^2)\big)$. As a result

$$f_R \approx \left(1 - \frac{v}{c}\right) f_E\,, \tag{54}$$

which indeed is the well-known classical expression. In order to get a general expression for the Doppler effect, we treat the radial component of the observers' velocity v_r separately by replacing v in the nominator of (52) by v_r. Finally we get

$$f_R = \frac{\sqrt{1 - \left(\frac{v}{c}\right)^2}}{1 + \frac{v_r}{c}}\, f_E\,. \tag{55}$$

This formula includes the transversal effect ($v_r = 0$) as a special case.

Not only the (classical) Doppler effect but other (classical) effects of signal propagation too have to be reformulated in the framework of special relativity. As an example, the final expressions for the aberration of light show some modifications due to special properties in the addition of velocities. Applying the Lorentz transformation to the relation between the observers' moving frame and the signal source's rest frame alters the classical expression for the aberration angle from $\tan\vartheta = v/c$ (e.g. Seidelmann/Fukushima [495], Schödlbauer [485]) to

$$\tan\vartheta = \frac{v}{c\sqrt{1-\left(\frac{v}{c}\right)^2}}\,. \tag{56}$$

29.7 Gravity and space-time metric

Adding the concept of force, we pass from special to general relativity. Inertial systems, as applied in special relativity, are systems, where strictly no forces do exist at all. In order to account for gravity, we distinguish real forces of attraction (gravitation) and apparent forces, e.g., due to (Earth) rotation, i.e. centrifugal forces. Following the weak equivalence principle (gravitational mass equals inertial mass), one can equate Newton's law of gravitation and his law of inertia to set up the classical equation of motion of a particle in case of its gravitational attraction by a primary body. Doing so, we equate the concepts of force and acceleration, knowing that gravitational attraction, rotation and acceleration are of different physical origin. The gravitational field, as described by the gravitational potential V, can be removed (locally) by a change of the coordinate system.

Considering (plane and uniform) rotations, e.g., with angular velocity ω in the xy-plane, the metric in an inertial frame $(\overline{x},\overline{y},\overline{z},\overline{t})$ changes from

$$\mathrm{d}s^2 = \mathrm{d}\overline{x}^2 + \mathrm{d}\overline{y}^2 + \mathrm{d}\overline{z}^2 - c^2\mathrm{d}\overline{t}^2 \tag{57}$$

due to the simple transformation

$$\begin{aligned}
\overline{x} &= \cos(\omega t)\,x - \sin(\omega t)\,y\\
\overline{y} &= \sin(\omega t)\,x + \cos(\omega t)\,y\\
\overline{z} &= z\\
\overline{t} &= t
\end{aligned} \tag{58}$$

into a metric with respect to a non-inertial frame (x,y,z,t)

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\left(1 - \frac{\omega^2\left(x^2+y^2\right)}{c^2}\right)\mathrm{d}t^2 - 2\omega y\mathrm{d}x\mathrm{d}t - 2\omega x\mathrm{d}y\mathrm{d}t. \tag{59}$$

The metric tensor now reads (reintroducing index notation again, i.e. $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$)

$$[g_{ij}] = \begin{bmatrix} -\left(1 - \dfrac{\omega^2\left(x_1^2+x_2^2\right)}{c^2}\right) & -\dfrac{\omega}{c}x_2 & +\dfrac{\omega}{c}x_1 & 0\\ -\dfrac{\omega}{c}x_2 & 1 & 0 & 0\\ +\dfrac{\omega}{c}x_1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{60}$$

The centrifugal potential is well-known to be $\Phi = \omega^2(x_1^2 + x_2^2)/2$. Thus, the component g_{00} can be written as $g_{00} = -1 + 2\Phi/c^2$. The components of the Coriolis force are related to the entries g_{0i} $(i = 1,2,3)$ of the metric tensor. Remark: if the rotation is non-uniform then the metric tensor becomes explicitly time-dependent too.

In the presence of gravitation, we add the corresponding field V to Φ, and simply replace the centrifugal potential by the gravity potential $W = V + \Phi$. On the other hand, in the absence of rotation, $W = V$ and the Coriolis force components of the metric tensor become zero, i.e. $g_{0i} = 0$ $(i = 1,2,3)$. The before mentioned statements are valid for weak fields, which is practically true in most cases, e.g., within the solar system. In case of strong fields, e.g., in the neighborhood of ultra-massive bodies like black holes, or for extremely precise applications, higher order terms had to be factored in any computations, because several effects are interrelated. Earth's rotational field is only partially represented by Φ (Moritz/Hofmann-Wellenhof [369]).

Now, V and Φ are already incorporated in the space-time metric and therefore do not explicitly appear as forces anymore. Rather, the motion of a particle follows a force-free geodesic. The use of the geodesic equation (14) is the price one has to pay for the use of non-inertial, e.g. rotating Earth-fixed, coordinates. Thus, the straight line from flat space-time is replaced by a general geodesic in curved space-time. On first sight, it is impossible to tell, whether a given metric refers to a flat or curved space-time. We can judge about this issue only after computing the Riemannian curvature tensor which will be related to the second derivatives of V, as we have seen before.

The equation of motion in Einstein's theory of gravitation, where gravitational forces are fictitious just as inertial forces and can be transformed away, result as an effect of curved space-time. In case of a non-rotating weak gravitational field it is governed by the metric tensor g_{ij} with $g_{00} = -1 + 2V/c^2$, $g_{0i} = 0$ and $g_{ij} = \delta_{ij}$ for $i = 1, 2, 3$. For a detailed analysis of this case we only need the equations (14), (15), (16), (29), (30), and (44). To start with the Christoffel symbols Γ_{kij}, only the Γ_{00j} , Γ_{k00} and Γ_{0i0} are potentially non-zero, because all g_{ij} other than g_{00} are constants. We find

$$\Gamma_{00j} = \frac{1}{2}\frac{\partial g_{00}}{\partial x_j}\,, \quad \Gamma_{k00} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x_k}\,, \quad \Gamma_{0i0} = \frac{1}{2}\frac{\partial g_{00}}{\partial x_i}\,, \tag{61}$$

and subsequently

$$\Gamma^0_{0j} = \Gamma^0_{j0} = -\frac{1}{c^2}\frac{\partial V}{\partial x_j}\,, \quad \Gamma^k_{00} = -\frac{1}{c^2}\frac{\partial V}{\partial x_k}\,. \tag{62}$$

To illustrate the validity of Einstein's geometrical interpretation of gravity we look at the geodesic equations for the spatial components ($i = 1, 2, 3$):

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}s^2} + \Gamma^i_{00}\left(\frac{\mathrm{d}x_0}{\mathrm{d}s}\right)^2 = 0\,. \tag{63}$$

For $v \ll c$ equations (49) and (50) yield $\mathrm{d}s \approx ic\mathrm{d}t = i\mathrm{d}x_0$, such that

$$\frac{\mathrm{d}x_0}{\mathrm{d}s} \approx \frac{1}{i} \quad \Rightarrow \quad \left(\frac{\mathrm{d}x_0}{\mathrm{d}s}\right)^2 \approx -1,\ \frac{\mathrm{d}^2 x_0}{\mathrm{d}s^2} \approx 0\,. \tag{64}$$

Therefore

$$\frac{\mathrm{d}^2 x_i}{-c^2\mathrm{d}t^2} - \frac{1}{c^2}\frac{\partial V}{\partial x_i}(-1) \approx 0 \tag{65}$$

and after multiplication with $-c^2$

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} \approx \frac{\partial V}{\partial x_i} \quad \Leftrightarrow \quad \ddot{\mathbf{x}} \approx \mathrm{grad}V, \tag{66}$$

which shows, that the classical equation of motion is equivalent to a Newtonian approximation of the more general relativistic equation of motion. As Moritz/Hofmann-Wellenhof [369] put it: the equation of a geodesic in a gravitationally curved space-time gives the Newtonian equation of motion in a „real“ gravitational field. It is also possible to derive the Poisson equation $\Delta V = -4\pi G\rho$ or the Laplace equation $\Delta V = 0$ from Einstein's theory by means of the Riemannian curvature tensor. In order to do so, it is of advantage to switch to the explicit notation of time, namely $x_0 = ct$, such that

$$\mathrm{d}s^2 = -\left(1 - \frac{2V}{c^2}\right)c^2\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{67}$$

and thus

$$g_{00} = -\left(1 - \frac{2V}{c^2}\right) \quad \Rightarrow \quad g^{00} \approx -\left(1 + \frac{2V}{c^2}\right). \tag{68}$$

In search for the classical expression we may neglect terms of order $O(c^{-2})$. Consequently, we solely retain Christoffel symbols $\Gamma^k_{00} = -\partial V/\partial x_k \neq 0$ for $k = 1, 2, 3$, and the Riemannian curvature tensor, i.e., its only non-zero components, becomes

$$R^i_{0k0} = -R^i_{00k} = \frac{\partial \Gamma^i_{00}}{\partial x_k} = -\frac{\partial^2 V}{\partial x_i \partial x_k}\,. \tag{69}$$

It already shows the fundamental importance of the 2nd order gravitational gradients. Measuring it by means of gradiometric techniques, in accordance with Marussi's theory of intrinsic geometry of the gravity field, is equivalent to measure the elements of the curvature tensor. The geometry of Earth's gravity field is completely determined by those 2nd order gradients, whereas the complete determination of space-time curvature requires the additional consideration of the influence of a centrifugal potential.

Equation (67) already implies that three-dimensional space remains flat, i.e. Euclidean, if only gravitational attraction is considered. The occurrence of the zeros in the sub-indices within (69) indicates the evolvement of time. Because all other components are zero, especially the pure spatial ones, the curvature of space-time is due to the time element and can fully be described by the Eötvös tensor of gravity gradients. Inclusion of gravitation changes the formerly flat space-time of special relativity into a curved space-time of general relativity. Gravitation, and hence gravity, is characterized by a non-vanishing curvature tensor. Conversely, fictitious gravitational forces are regarded as being caused by curvilinear coordinates in a curved space-time, where curvature is induced by the presence of mass. In this sense, a satellite orbit in space can be viewed as a projected geodesic from space-time into space.

Fictitious inertial forces are caused by accelerated/rotating systems in a flat space-time. Like gravitation, they correspond to curvilinear coordinate systems, but only gravitation leads to a non-vanishing curvature tensor. If it vanishes, we are sure about the absence of gravitational forces. In this case one can introduce a truly inertial coordinate system or use an accelerated and rotating coordinate system, R^i_{jkl} will remain zero for all index combinations. If $R^i_{jkl} \neq 0$ then it is not possible to set up a globally valid inertial system, and gravitation could only be removed locally.

29.8 Einstein field equations

Neglecting the centrifugal potential, the Eötvös tensor will be denoted by $V_{ij} = \partial^2 V/\partial x_i \partial x_j$. In empty space, i.e. $\rho = 0$, the Laplace equation $\Delta V = 0$ can be written with use of the contracted Eötvös tensor as $V_{ii} = 0$. Alternatively, following the above statements, we can replace the Eötvös tensor by the Riemannian curvature tensor. Again applying the contraction of a tensor we get the Ricci curvature tensor $R_{jk} = R^i_{jki}$ and get Einstein's equivalent to the Laplace equation. In empty space we have

$$R_{jk} = 0, \tag{70}$$

which can even be contracted further, leading to the Ricci scalar R, by

$$g^{jk} R_{jk} = R^j_j =: R. \tag{71}$$

Using these relations one can introduce the Einstein tensor

$$G_{ij} := R_{ij} - \tfrac{1}{2} R g_{ij} \tag{72}$$

to finally set up the Einstein field equations (assuming the cosmological constant as being zero) (Stephani [524])

$$G_{ij} = \kappa T_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij}, \tag{73}$$

where κ is a constant of nature ($\kappa = 8\pi G/c^4 \approx 2.077 \cdot 10^{-43} s^2 kg^{-1} m^{-1}$ in case of Newtonian approximation), sometimes called Einsteinian [gravitational] constant, and T_{ij} represents the so-called energy-momentum tensor (Heitz/Stöcker-Meier [240]). The latter characterizes matter and generalizes the mass density ρ.

One distinguishes ρ with respect to a co-moving frame from its invariant counterpart in a barycentric frame, i.e. the baryon mass density (also known as Fock density of matter) $\rho^* = \sqrt{-g}\, u^0 \rho$ with $g = \det(g_{ij})$, which satisfies a so-called equation of continuity (Kopejkin et al. [299])

$$c\rho^*_{,0} + \left(\rho^* v^i\right)_{,i} = 0, \tag{74}$$

where $v^i = cu^i/u^0$ is the three-dimensional velocity of matter. The original continuity equation itself is a direct consequence of the stated conservation of the energy-momentum tensor, i.e. $\nabla_\nu T^{\mu\nu} = 0$, such that

$$\nabla_\alpha \left(\rho u^\alpha\right) = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\rho \sqrt{g}\, u^\alpha\right) = 0. \tag{75}$$

The energy-momentum tensor comprises mechanical, thermal, and electromagnetic parts, each of which can be separated further into different subparts. For example, the mechanical part contains kinematical, potential, and viscous terms (Börger [62]).

Introducing an internal energy per unit mass u one can relate T_{00} to the energy density, namely $T_{00} = \mu c^2$, where $\mu = \rho(1 + u/c^2)$ represents the mass density. Einstein's equations generalize the Poisson equation of the Newtonian theory of gravity and the metric tensor extends the notion of the Newtonian gravitational potential (Kopejkin et al. [299]).

Einstein's equation for empty space results in

$$G_{ij} = 0 = R_{ij} - \tfrac{1}{2} R g_{ij}, \tag{76}$$

which is satisfied if $R = 0$ and $R_{ij} = 0$ is simultaneously true. $R_{ij} = 0$ means that space-time is as flat as possible. It does not imply $R^i_{jkl} = 0$, as already mentioned before, which is a much stronger condition and stands for complete flatness in all of the n dimensions.

Equivalently, by applying the variational principle, i.e. by taking the variational derivative of a problem-dependent given Lagrangian $\mathcal{L}$ based action equation (Hehl [238]) with respect to the metric tensor and its spatial derivatives (Kopejkin et al. [299]), one could write, again using a contracted quantity $T = T_i^i$, i.e. $T = g^{ij}T_{ij}$,

$$R_{ij} = \kappa\big(T_{ij} - \tfrac{1}{2}Tg_{ij}\big). \tag{77}$$

Following the laws of conservation, the energy-momentum tensor of matter itself satisfies a variational equation:

$$\frac{1}{2}\sqrt{-g}\,T_{ij} = \frac{\partial\big(\sqrt{-g}\,\mathcal{L}\big)}{\partial g^{ij}} - \frac{\partial}{\partial x^k}\frac{\partial\big(\sqrt{-g}\,\mathcal{L}\big)}{\partial\big(\partial_k g^{ij}\big)}\,. \tag{78}$$

As a remark, in case of alternative (more general) theories of gravitation, e.g. in scalar-tensor theories comprising the additional action of a scalar field (leading to an eleventh field equation), one may apply a conformal transformation of the metric tensor and introduce a corresponding conformal Ricci tensor to finally get field equations in a form that is as simple as possible. Einstein's theory of general relativity (d'Inverno [112]) can be regarded as a special (limiting) case with constant and thus unobserved scalar field. Scalar field modes are related, for instance, to the alleged phenomenon of gravitational wave emission.

The metric tensor g_{ij} can be regarded as a solution of Einstein's field equations with imposed boundary and initial conditions. The usage of boundary conditions is physically motivated by the application of the principle of causality (strong, i.e. practical, version: similar causes lead to similar effects versus weak, i.e. impractical, version: equal causes lead to equal effects) to gravitational fields. The g_{ij} is used in real measurements of time intervals and spatial distances and therefore called a physical metric, whereas a conformal metric is called the Einstein-frame metric which may be more convenient for doing calculations (Kopejkin et al. [299]). Nevertheless one usually prefers the (non-conformal) metric tensor for the sake of an easier meaningful physical interpretation of final results.

Applying our simple weak gravitational field example from above leads after contraction of (69) to

$$R^i_{00i} = R_{00} = \frac{\partial^2 V}{\partial x_i \partial x_i} = \Delta V \quad \Rightarrow \quad \Delta V = 0 \Leftrightarrow R_{00} = 0, \tag{79}$$

or, in case of a non-empty space, we get the classical Newtonian field equation

$$\Delta V = R_{00} = -4\pi G\rho, \tag{80}$$

again showing the relevance of the pure time component of the curvature tensor. The Laplace operator in Euclidean space is defined as $\Delta := \delta^{ij}\partial_i\partial_j$ for $i,j = 1,2,3$. For details on the derivation of the Poisson equation see for instance Stephani [524]. To summarize, the Newtonian theory of gravitation can be viewed as a special case or limit of Einstein's theory of gravitation, namely for weak gravitational fields ($V \ll c^2$) and slow motions ($v^2 \ll c^2$). Remark: instead of a post-Newtonian approximation scheme in case of slow motions one should apply a so-called post-Minkowskian approximation scheme (Kopejkin et al. [299]) if fast moving bodies are involved. Rendall [452] provides some details on the justification and breakdown of (higher order) post-Newtonian approximations which relates to the (non-)existence of several interdependent small parameters. The post-Minkowskian scheme solves the field equations in terms of retarded gravitational potentials (the retardation effect simply shows up due to the limited speed of propagation of gravity).

29.9 Special case: Schwarzschild metric and resulting testable relativistic effects

The Schwarzschild metric is one of the few special cases for which an exact solution of Einstein's field equations has been found (Stephani et al. [525]), even though many exact solutions (Bičák [43]) exist for the special case $T_{ij} = 0$. In order to illustrate and get used to the calculation of a specific Riemann tensor and its implications, as an aside, we explicitly provide the intermediate steps in the following.

Representing the spherically symmetric gravitational attraction of a point mass M it is given in usual spherical coordinates $(r, \theta, \lambda; t)$ by (Schwarzschild [493], Schneider [481])

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\lambda^2 \tag{81}$$

with $m = GM/c^2$ having the character of a mass, but physical dimension of a length. Introducing an auxiliary quantity $\mu(r) := 1 - 2m/r$ leads to the only non-zero Christoffel symbols

$$\begin{aligned} &\Gamma_{001} = \Gamma_{010} = -\Gamma_{100} = \frac{1}{2}\frac{\partial\mu}{\partial r}\,, \quad \Gamma_{111} = \frac{1}{2\mu^2c^2}\frac{\partial\mu}{\partial r}\,, \quad \Gamma_{122} = -\Gamma_{212} = -\Gamma_{221} = \frac{r}{c^2}\,, \\ &\Gamma_{133} = -\Gamma_{313} = -\Gamma_{331} = \frac{r\sin^2\theta}{c^2}\,, \quad \Gamma_{233} = -\Gamma_{323} = -\Gamma_{332} = \frac{r\sin\theta\cos\theta}{c^2} \end{aligned} \tag{82}$$

and

$$\begin{aligned}
&\Gamma^0_{01} = \Gamma^0_{10} = -\Gamma^1_{11} = \frac{1}{2\mu}\frac{\partial \mu}{\partial r}\,, \quad \Gamma^1_{00} = \frac{\mu c^2}{2}\frac{\partial \mu}{\partial r}\,, \quad \Gamma^1_{22} = -\mu r\,, \Gamma^1_{33} = -\mu r \sin^2\theta\,,\\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}\,, \quad \Gamma^2_{33} = -\sin\theta\cos\theta\,, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta\,.
\end{aligned} \tag{83}$$

The following non-zero components of the Riemannian curvature tensor do exist

$$\begin{aligned}
R^1_{001} = -2R^2_{002} = -2R^3_{003} &= \frac{2mc^2}{r^3}\left(1-\frac{2m}{r}\right) = -R^1_{010} = 2R^2_{020} = 2R^3_{030}\,,\\
R^0_{110} = -2R^2_{112} = -2R^3_{113} &= -\frac{2m}{r^2(r-2m)} = -R^0_{101} = 2R^2_{121} = 2R^3_{131}\,,\\
-2R^0_{220} = -2R^1_{221} = R^3_{223} &= -\frac{2m}{r} = 2R^0_{202} = 2R^1_{212} = -R^3_{232}\,,\\
-2R^0_{330} = -2R^1_{331} = R^2_{332} &= -\frac{2m}{r}\sin^2\theta = 2R^0_{303} = 2R^1_{313} = -R^2_{323}\,.
\end{aligned} \tag{84}$$

Contraction yields only zero-valued components of the Ricci curvature tensor:

$$R^k_{00k} = R_{00} = 0\,, \quad R^k_{11k} = R_{11} = 0\,, \quad R^k_{22k} = R_{22} = 0\,, \quad R^k_{33k} = R_{33} = 0\,. \tag{85}$$

Setting up the individual geodesic equations we find a coupled system of four ordinary differential equations of second order

$$\begin{aligned}
&\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \frac{m}{\mu r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \mu r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 - \mu r\sin^2\theta\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 + \frac{\mu m c^2}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0\,,\\
&\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 = 0\,,\\
&\frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2} + 2\left(\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau} + \cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = 0\,,\\
&\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{2m}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} = 0\,.
\end{aligned} \tag{86}$$

where the already given setting $s = ic\tau \Rightarrow \mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$ has been applied. Several books discuss the solution of this equation system (e.g. Moritz/Hofmann-Wellenhof [369], Schneider [481]). The derivation yields several integrals of motion which are well-known from classical celestial mechanics, i.e. the solution of the idealized two-body problem (Kepler problem). The final solution will take a similar form. Besides the integrals of motion we seek to find a final expression for the evolution of the radial distance r of the secondary body (e.g. a satellite) with respect to the primary body (e.g. the Earth).

The classical solution could be regarded as an approximate solution to the relativistic problem, thus

$$r_0 = \frac{p}{1 + e\cos f}\,, \tag{87}$$

being the general equation of a conic section with the shape governed by the numerical eccentricity e, the size defined by semi-major axis a or semi-latus rectum $p = a(1-e^2)$, and the position of the secondary body within its orbit given by the true anomaly f. The resulting relativistic orbit equation shows a slight but essential modification:

$$r = \frac{p}{1 + e\cos\left(\left(1 - \dfrac{3GM}{c^2 p}\right)\lambda\right)}\,. \tag{88}$$

Considering point masses we may choose without loss of generality any orbital plane, e.g. the one identical to Earth's equatorial plane. Therefore, $\mathrm{d}f = \mathrm{d}\lambda$ and we can use λ as our new independent variable, equally counting from the periapsis per definition. It is important to note that the period for the relativistic case is now larger than 2π because

$$P = \frac{2\pi}{1 - \dfrac{3GM}{c^2 p}} \approx 2\pi\left(1 + \frac{3GM}{c^2 p}\right), \tag{89}$$

where we neglect terms of higher order ($O(c^{-4})$). Obviously, there is a relativistic shift of the periapsis:

$$\Delta\lambda = P - 2\pi \approx \frac{6\pi GM}{c^2 a(1-e^2)}\,. \tag{90}$$

This shift had to be superimposed on the classical precessional effects which come into play when considering a non-spherical primary body. The relativistic precession already applies to point masses. From equation (90) follows $\Delta\lambda \propto 1/a$. The closer the secondary body to the primary body, the more significant the relativistic shift of the periapsis becomes. Exemplarily applying this formula to planetary motion within the solar system, we recognize that planet Mercury will exhibit the largest relativistic deviation from its classical solution due to the mass of the Sun, i.e., its influence on the curvature of space-time. Taking nominal values for the astronomical quantities of Mercury (a_M, e_M), the Sun ($M_\odot$), and for the fundamental quantities (c, G) (Cox [108]) one gets

$$\Delta\lambda_M \approx 0.103516'' \, \frac{1}{\text{rev}} \approx 42.98'' \, \frac{1}{\text{century}} \,. \tag{91}$$

Of course, one rather uses the more precise gravitational parameter of the Sun $\mu_\odot = GM_\odot$ instead of the product of the single quantities G and $M_\odot$. This is a tiny but significant effect that, in principle, has to be accounted for also in highly precise satellite orbit calculations or ephemeris computations.

Another relativistic effect, besides the gravitational time delay (relativistically generalized Doppler effect) or relativistic correction of the precessional motion in the two-body problem, is the deflection or bending of an electromagnetic signal, e.g. a light ray, due to gravitational attraction. Remark: for highly precise applications it is no longer sufficient to treat the deflecting body as a mass monopole (Zschocke/Klioner [614]).

For a light-like geodesic with $\mathrm{d}s^2 = 0$ we also have $\mathrm{d}\tau = 0$, which has to be accounted for in the equations (86). Again solving the resulting system approximatively, one gets an expression for the minimal distance D of a light ray passing a massive body (Moritz/Hofmann-Wellenhof [369])

$$D = r\cos\lambda + \frac{GM}{c^2 D}\big(r + r\sin^2\lambda\big). \tag{92}$$

An auxiliary transformation from polar coordinates (r, λ) into cartesian coordinates (x, y), where the x-axis originates at the center of the massive body and points towards the point of closest approach of the signal, enables the straightforward derivation of a formula for the small bending angle δ of the light ray. It finally reads

$$\tan\delta \approx \delta = \frac{4GM}{c^2 D} \,. \tag{93}$$

As an example, we consider a light ray passing the Sun right at its rim. With values $D = R_\odot \approx 695000\,km$ and $GM_\odot = 132712440018\,km^3/s^2$ one gets $\delta = 1.753''$, which was verified by astronomers through observations. Einstein's theory of gravitation also predicts several additional bending effects that are of second order caused by higher order post-Newtonian terms of the spherical solar field ($\approx 11 \cdot 10^{-6}\,''$), solar angular momentum ($\approx 0.7 \cdot 10^{-6}\,''$), Sun's oblateness ($\approx 0.2 \cdot 10^{-6}\,''$) (Soffel [511]), or even Jupiter's influence (Soffel et al. [512]). The Schwarzschild metric (taking only mass into account) provides a first mathematical basis for studies of black hole phenomena. If additional properties (besides mass) shall be considered, one has to apply another metric, e.g., the Kerr(-Newman) metric which comprises mass and angular momentum (and electric charge).

29.10 Inertial systems and general relativity

In relativistic geodesy, we are more interested in the impact of special and general relativistic effects on genuine geodetic tasks and observations. In geodetic practice one often introduces the concept of (quasi-)inertial systems. Even in the context of general relativity those systems can be established locally to a very good approximation by neglecting terms of order $O(c^{-2})$. Nonetheless there exist differential, i.e. tidal, gravitational forces in such a system, because it is not possible to exactly remove or transform away gravitational effects other than in a point-wise manner. These tidal forces correspond to the above mentioned 2nd order gradients, i.e. the Riemannian curvature tensor, and are measurable in satellite geodesy by space-spaced gradiometers whereas on-board gravimeters would indicate zero measurements for gravity itself due to point-wise weightlessness.

In opposition to the rigorous sense of Einstein's theory there do exist privileged systems in practice which are approximatively inertial, because space-time is asymptotically flat. The adjective „inertial" may refer to translation and/or rotation. Inertial systems are always at the state of uniform motion with respect to each other which implies relative motion on a straight line with constant velocity. In geodesy, most three-dimensional cartesian coordinate systems, e.g. ECI (Earth Centered Inertial), are inertial only with respect to rotation. Its orientation remains fixed against the positions of remote stars but the origin moves non-uniformly or arbitrarily in space. The description of the non-uniform motion of a quasi-inertial system makes use of the concept of Fermi propagation, i.e. the Fermi-Walker transport (Stephani [523]). It is a generalization of the parallel transport, because it is valid for the transport of a vector (or tetrad-system) along an arbitrary curve, whereas the usual parallel transport is limited to world lines that are geodesics.

An observer moving along an arbitrary world line $x^i(\tau)$ can regard himself being at rest and his spatial axes as non-rotating, and hence inertial, if he chooses an appropriate local coordinate system accounting for the possible action of forces. He will carry along a tetrad system such that any transported vector shows constant orientation with respect to the tangent vector $\mathrm{d}x^i/\mathrm{d}\tau$ of his path, because being at rest means that the four-velocity v^i does not possess any non-zero spatial component. This observer would regard all vectors as constant that do not change with respect to his moving local coordinate system. Comprising the intrinsic differential (23), the Fermi-Walker transport of a (contravariant) vector λ^i is defined by the Fermi differential

$$\mathrm{D}\lambda^i = \lambda_j \left(\mathrm{D}v^j\, v^i - \mathrm{D}v^i\, v^j\right) = \lambda_j \mathrm{D}v^j\, v^i - \mathrm{D}v^i\, \lambda_j v^j . \tag{94}$$

In case of a four-vector λ^i with zero-valued time component and a corresponding four-velocity v^j with zero-valued spatial components, which means resting in a local system without velocity in rest space, we have $\lambda_j v^j = 0$ and therefore

$$\frac{\mathrm{D}\lambda^i}{\mathrm{D}s} = \lambda_j \frac{\mathrm{D}v^j}{\mathrm{D}s}\, v^i . \tag{95}$$

The change $\mathrm{D}\lambda^i/\mathrm{D}s$ has the direction v^i and thus no component in the rest space of the observer. Any change occurs purely in time, and λ^i remains unchanged in space. Therefore, Fermi propagation is closely related to spatial parallelism, but small precessional effects may exist (Moritz/Hofmann-Wellenhof [369]). Combining three mutually orthogonal vectors that conjointly are subject to Fermi-Walker transportation, one can practically realize an approximative quasi-inertial system in the framework of general relativity. As an example, consider the gyroscopes within inertial navigation systems. As we have seen, (quasi-)inertial systems may be defined either inertially, e.g. by means of gyroscopic motion, or astronomically, e.g. by means of light ray propagation. This was the reason for Weyl [577] to introduce the terms inertial compass and stellar compass, respectively.

29.11 Geodesic deviation equation

Relativistic geodesy, comprising the use of precise atomic clocks, relies on the accurate modeling of the gravity field. Considering the world lines of atomic clocks, i.e. its time-like geodesics in space-time, also requires the determination of its potential deviations due to tidal forces. We recognized already that this effect is related to the Riemannian curvature tensor.

Tidal forces acting upon atomic clocks, e.g. located at Earth's surface, are induced by the gravitational attraction of disturbing masses (e.g. Sun, Moon, and planets). Each mass will contribute to the resulting space-time curvature and thus to the overall gravitational potential value at the actual position of the observer (atomic clock). An earthbound clock is co-moving with Earth's center of mass, both being neighboring points in space-time. To a good approximation the paths of these points can be regarded as geodesics that will deviate due to the existence of tidal forces. With $\xi^i(\tau)$ denoting the deviation vector we set up the general geodesic equation (14) for the two world lines $x^i(\tau)$ and $x^i(\tau) + \xi^i(\tau)$ separately. The second equation contains a term $\Gamma^i_{jk}(\mathbf{x} + \boldsymbol{\xi})$ which can be rewritten by expanding the Christoffel symbol in a Taylor series about the point $\mathbf{x}$ with deviation $\boldsymbol{\xi}$ and neglecting higher order terms. Subtracting both geodesic equations results in a fundamental equation of the geodesic deviation, again deploying the intrinsic derivation for the sake of simplification (Nieto et al. [391]):

$$\frac{\mathrm{D}^2\xi^i}{\mathrm{D}\tau^2} + R^i_{jkl}\, \xi^k \frac{\mathrm{d}x^j}{\mathrm{d}\tau} \frac{\mathrm{d}x^l}{\mathrm{d}\tau} = 0 . \tag{96}$$

To find a Newtonian approximation for this geodesic deviation, a couple of further simplifying assumptions are made. First we introduce a quasi-inertial system by using free-fall coordinates. This results in $\Gamma^i_{jk} = 0$, and hence the intrinsic 2nd derivative will be equal to the ordinary 2nd derivative. Then, in order to get proper relations for the orders of magnitudes of the metric tensor components, we reintroduce t as time coordinate such that $x_0 = t$ and therefore $\mathrm{d}/\mathrm{d}\tau = \mathrm{d}/\mathrm{d}t$. Furthermore, we assume deviations only in space, i.e. $\xi^0 = 0$, and remember equation (69) which means that the only non-zero components of the Riemannian curvature tensor to be accounted for can be written as $R^i_{00j} = \partial^2 V/\partial x_i \partial x_j$. One finally gets the geodesic deviation in Newtonian approximation as (Moritz/Hofmann-Wellenhof [369])

$$\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2} = \frac{\partial^2 V}{\partial x_i \partial x_j}\, \xi_j , \tag{97}$$

where we switched from upper to lower indices. This result is in full agreement with classical mechanics; we accordingly apply Newton's law $\mathrm{d}^2 x_i/\mathrm{d}t^2 = \partial V/\partial x_i$ to the neighboring point. Thus, we replace the argument by $x_i + \xi_i$ and expand the right hand side, i.e. $\partial V/\partial(x_i + \xi_i)$, in a Taylor series again. Neglecting higher order terms results in equation (97), the left hand side of which can be interpreted as a differential acceleration and the right hand side as a tidal, i.e. differential, force acting upon a unit mass, respectively.

Exemplarily, we treat the Moon as a disturbing point mass with the lunar potential simply given by $V_l = GM_l/r_l$, where r_l is the radial distance from this point mass position. The 1st and 2nd order partial differentials are

$$\frac{\partial V_l}{\partial x_i} = -GM_l \frac{x_i}{r_l^3}\,, \quad \frac{\partial^2 V_l}{\partial x_i \partial x_j} = -GM_l \frac{r_l^3 \delta_{ij} - 3x_i r_l^2 \dfrac{x_j}{r_l}}{r_l^6}\,. \tag{98}$$

From the second equation in (98) we deduce the validity of Laplace's equation

$$\frac{\partial^2 V_l}{\partial x_i^2} = -GM_l \frac{r_l^2 - 3x_i^2}{r_l^5} \quad \Rightarrow \quad \Delta V_l = 0 \tag{99}$$

and find an explicit expression for the tidal force

$$f_i := \frac{\partial^2 V_l}{\partial x_i \partial x_j}\,\xi_j = -GM_l \left(\frac{\xi_i}{r_l^3} - \frac{3x_i x_j \xi_j}{r_l^5} \right). \tag{100}$$

The latter formula may be simplified by the choice of a special (orthogonal) coordinate system. In order to do so, we place its origin at Earth's center of mass, and let the x_3-axis point towards the lunar point mass, to get the position vector of the Moon as $\mathbf{x}_l = (0, 0, r_l)^T$.

Now introduce spherical coordinates such that a position on the surface of the Earth (assumed to be spherical with radius $R_\oplus$), e.g. an atomic clock observation site A representing the neighboring point, is defined by

$$\boldsymbol{\xi}_A = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} R_\oplus \sin\theta_A \cos\lambda_A \\ R_\oplus \sin\theta_A \sin\lambda_A \\ R_\oplus \cos\theta_A \end{pmatrix}, \tag{101}$$

where the angles do not refer to the usual geographical coordinate system. Instead, polar distance θ here refers to the direction towards the Moon, and λ acts as a longitude-like angle within the plane perpendicular to that direction.

Introducing the above mentioned special choice for the lunar position vector into equation (100) yields the components of the tidal force acting upon A

$$\mathbf{f}_A = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{GM_l}{r_l^3} \begin{pmatrix} -\xi_1 \\ -\xi_2 \\ 2\xi_3 \end{pmatrix}, \tag{102}$$

which can be rewritten as the gradient of the following potential function:

$$V = \frac{GM_l}{r_l^3} \left(-\tfrac{1}{2}\xi_1^2 - \tfrac{1}{2}\xi_2^2 + \xi_3^2 \right) \quad \Rightarrow \quad \mathbf{f}_A = \begin{pmatrix} \partial V/\partial f_1 \\ \partial V/\partial f_2 \\ \partial V/\partial f_3 \end{pmatrix}. \tag{103}$$

Replacing the cartesian components ξ_i by means of equation (101) leads to

$$V = \frac{GM_l}{r_l} \left(\frac{R_\oplus}{r_l} \right)^2 P_2(\cos\theta_A)\,, \tag{104}$$

where $P_2(\cos\theta) = (-1 + 3\cos^2\theta)/2$ is a Legendre polynomial of degree 2. Equation (104) represents the classical tidal potential formula of a disturbing (point) mass. Analogous expressions have to be set up for any other significant third bodies, depending on the required accuracy, e.g., the Sun and major planets may be included.

As we have seen, tidal forces even remain in a free falling system. They act as residual forces that can be recognized in weightlessness via detection of differential changes by means of gradiometer measurements or atomic clock readings. As indicated before, gravitational forces only can be removed point-wise by an opposite inertial acceleration field. At that very point we get $\Gamma^i_{jk} = 0$ but in the neighborhood we have $\Gamma^i_{jk} \neq 0$, though these non-zero values may be very small. In principle, we should be able to determine the gravitational field, i.e. the Riemannian curvature tensor, by highly precise geodetic measurements of the (changing) length of the deviation vector ξ^i and/or by clock reading comparisons. For example, Shirokov [504] applied the geodesic deviation equation (96) to the Schwarzschild metric in order to set up a gradiometer scenario comprising two neighboring test masses orbiting a black hole. In (physical) geodesy, the geodesic deviation equation is the theoretical basis of relativistic gradiometry (Paik [400]).

Gravimeters only measure the resulting acceleration (gravitational plus inertial). Its decomposition requires the knowledge of more than a single force vector and is only possible due to the different structure of the gravitational and inertial field. For an extended and rigid moving system the (inertial) acceleration may be identical at any single point, but gravitation will differ slightly from point to point because of the existence of those tidal forces.

29.12 Separability of different kinds of forces

Moritz/Hofmann-Wellenhof [369] outline the general separability of gravitational and inertial effects in the framework of classical mechanics by comparing the Newtonian equations of motion with respect to an inertial system (denoted by capital letters)

$$\ddot{X}_i = F_i \tag{105}$$

against corresponding equations with respect to a moving system (denoted by small letters)

$$\ddot{x}_i = f_i + 2\omega_{ij}\dot{x}_j + \big(\dot{\omega}_{ij} + \omega_{ik}\omega_{jk}\big)x_j - \ddot{b}_i, \tag{106}$$

where both, translational motion and rotational motion is accounted for and may be time dependent. In index notation, the former is represented by a displacement vector B_i, whereas the latter is described by a rotation matrix a_{ij}, such that the transformation reads

$$X_i = a_{ij}x_j + B_i. \tag{107}$$

The skew-symmetric matrix $\omega_{jk} = -\omega_{kj}$ in equation (106) stems from products $\omega_{jk} = a_{ik}\dot{a}_{ij}$ and $\omega_{kj} = \dot{a}_{ik}a_{ij}$, because $\mathrm{d}(a_{ik}a_{ij})/\mathrm{d}t = \dot{\delta}_{kj} = 0$. Its non-diagonal elements define the instantaneous axis of rotation, expressed by the vector $\boldsymbol{\omega} = (\omega_{23}, \omega_{31}, \omega_{12})^T$.

The equation of motion for the moving system can also be brought into factitious Newtonian form by conflating the right hand side of equation (106), such that $\ddot{x}_i = f_i^*$. The total force f_i^* will by measured by means of accelerometry, where $f_i = \partial V/\partial x_i$ is the only gravitational force. All other terms belong to fictitious inertial forces, distinguishing between Coriolis force $2\omega_{ij}\dot{x}_j$, centrifugal force $\omega_{ik}\omega_{jk}x_j$, Euler force $\dot{\omega}_{ij}x_j$, and Einstein force $-\ddot{b}_i$ (Lanczos [317]). The Euler force accounts for a possibly non-uniform rotation.

The separability of different kinds of forces becomes obvious, if we resort the individual forces and take the first few gradients. Considering the case of a co-moving clock ($\dot{x}_j = 0$) we can neglect the Coriolis force and get the 1st order gradient

$$\frac{\partial V}{\partial x_i} = f_i^* - \big(\dot{\omega}_{ij} + \omega_{ik}\omega_{jk}\big)x_j + \ddot{b}_i. \tag{108}$$

The next higher order gradients follow as

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_i^*}{\partial x_j} - \big(\dot{\omega}_{ij} + \omega_{ik}\omega_{jk}\big), \quad \frac{\partial^3 V}{\partial x_i \partial x_j \partial x_k} = \frac{\partial f_i^*}{\partial x_j \partial x_k}\,. \tag{109}$$

Now, depending on the stabilization of our measuring platform, certain gradients will be affected by inertial effects or not. If our instrument is inertially stabilized, i.e., its x_i-axes defining the moving system are permanently kept parallel to the inertial X_i-axes, there are no rotations. That means $\boldsymbol{\omega} = 0$ or $\omega_{ij} = 0$, and therefore the corresponding 1st order gradient $\partial V/\partial x_i = f_i^* + \ddot{b}_i$ still contains some linear acceleration (inertial effect), but the 2nd order gradient $\partial^2 V/\partial x_i \partial x_j = \partial f_i^*/\partial x_j$ is already unaffected by any inertial effects. Thus, given the ideal case were we assume the absence of instrumental misalignments or any other systematic errors, integrated measurements of the 2nd order gradients by means of an inertially stabilized gradiometer will allow us to separate the individual kinds or forces. The purely gravitational Riemannian curvature tensor is not affected by inertial fields. Regarding higher order gradients, all inertial effects vanish completely whether inertial stabilization is used or not. On the other hand, by combination of gravimetry and gradiometry, separation becomes possible also for the 1st order gradients. These can either be regarded as a vector of gravitational force (in case of airborne or space-borne instrumentation) or as a vector of gravity (in case of ground-based instrumentation that is affected by the centrifugal potential, too). Combining a tensor gradiometer that measures the full Eötvös tensor $V_{ij} = \partial^2 V/\partial x_i \partial x_j$ and a vector accelerometer that simultaneously measures the total force vector f_i^*, one can finally set up a second order linear differential equation for the velocity $\dot{b}_i$ of the moving system

$$\frac{\mathrm{d}^2 \dot{b}_i}{\mathrm{d}t^2} - V_{ij}\dot{b}_j + \dot{f}_i^* = 0, \tag{110}$$

where the variable coefficients V_{ij} and $\dot{f}_i^*$ are given by simultaneous gradiometer and accelerometer measurements. Equation (110) may be solved by numerical integration. Knowing the inertial acceleration or Einstein force $\ddot{b}_i$ which is unaffected by gravitational effects, we can calculate the 1st order gradient $V_i = \partial V/\partial x_i = f_i^* + \ddot{b}_i$. Integration of the velocity $\dot{b}_i$ yields the position vector b_i of the moving system with respect to the (quasi-)inertial system.

29.13 Various relativistic effects

In order to account for the capabilities of available highly precise measurement techniques, e.g. (optical) atomic clock readings, one has to consider additional small relativistic effects that might be significant, namely gravitomagnetic effects. These are induced by non-zero components g_{0i} of the metric tensor, forming a space-vector $\mathbf{m} := (g_{01}, g_{02}, g_{03})^T$. Whereas in Newtonian mechanics the gravitational field of a sphere does not depend on its rotational status, in general relativity there exist a dragging effect which means that a rotating sphere would drag the surrounding space-time. Of course, the usual Coriolis force is also characterized by non-zero g_{0i}. This classical effect can be removed by proper choice of an inertial reference system, but the dragging effect would remain.

Taking into account higher order relativistic effects, the line element will change from equation (67) to

$$\mathrm{d}s^2 = \left(-1 + 2\left(\frac{V}{c^2}\right) - 2\left(\frac{V}{c^2}\right)^2\right)c^2\mathrm{d}t^2 + \left(1 + 2\left(\frac{V}{c^2}\right)\right)\mathrm{d}x_i\mathrm{d}x_i + 2g_{0i}\mathrm{d}x_i c\mathrm{d}t. \tag{111}$$

Moritz/Hofmann-Wellenhof [369] provide the corresponding formula for the more general PPN approximation, i.e. $\beta, \gamma \neq 1$.

Earth's gravitomagnetic field is represented by the vector $\mathbf{m}_\oplus$ via

$$\mathbf{m}_\oplus = -\frac{2GM_\oplus}{c^3 r^3}\,\mathbf{J}_\oplus \times \mathbf{x}, \tag{112}$$

where $\mathbf{J}_\oplus$ is the Earth's angular momentum vector per unit mass, which can be approximated by

$$\mathbf{J}_\oplus = \frac{I_\oplus \boldsymbol{\omega}_\oplus}{M_\oplus} \approx \begin{pmatrix} 0 \\ 0 \\ 0.33\,R_\oplus^2\,\omega_\oplus \end{pmatrix} =: J_\oplus \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{113}$$

The use of a factor 0.33 leads to a better approximation for the moment of inertia $I_\oplus$ of the Earth (approximated by a sphere of radius $R_\oplus$), than the classical formula $I = (2/5)Mr^2$ for a solid sphere.

Gravitomagnetic effects are of gyroscopic nature. Whereas in classical mechanics a rotating gyroscope remains its orientation when transported parallel to its rotational axis, the theory of general relativity predicts the existence of small measurable precessional effects (Schröder [487]), primarily the geodetic precession and the Lense-Thirring precession. Secondary precessional effects as well as spin-curvature coupling effects (Misner et al. [366]) are neglected here.

We assume parallel propagation without linear acceleration, i.e. a special case of the Fermi-Walker transport. As a remark, equivalent formulas result from a derivation based on the asymptotic matching technique (Kopejkin et al. [299]). The spatial angular momentum vector is being used to construct a four-dimensional spin vector S_i in a rest frame. It is a purely space-like vector with zero time component ($S_0 = 0$). A corresponding 4-velocity vector $v^i = \mathrm{d}x^i/\mathrm{d}\tau = (1, 0, 0, 0)^T$ results for the case of no translational motion in the rest frame. Therefore, both vectors satisfy the orthogonality condition $S_i v^i = 0$, which is valid not only in the rest frame.

Applying the above mentioned formulas for the parallel transport of contravariant vectors yields the timely evolution of the spin vector $S^i = g^{ij}S_j$ formally as

$$\frac{\mathrm{d}S^i}{\mathrm{d}t} = -\Gamma^i_{jk} S^j v^k, \tag{114}$$

where the tedious task of deriving the Christoffel symbols for the metric (111) arises (Weinberg [573]). We are mainly interested in the change of the spatial spin axis $\mathbf{s}$, which finally results in a simple precessional motion, i.e. infinitesimal rotation (Moritz/Hofmann-Wellenhof [369])

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \mathbf{s}, \tag{115}$$

governed by the angular velocity vector $\mathbf{\Omega}$ with

$$\mathbf{\Omega} = \frac{3}{2c^2}\,\mathbf{v}\times\nabla V_\oplus - \frac{c}{2}\,\nabla\times\mathbf{m}_\oplus =: \mathbf{\Omega}_{GP} + \mathbf{\Omega}_{LTP}. \tag{116}$$

The first term on the right hand side, called geodetic precession or de Sitter precession, is caused by the motion of a gyroscope through a (possibly static) curved space-time. Thus, it would also occur for a point mass or for a non-rotating sphere and can be written as

$$\mathbf{\Omega}_{GP} = \frac{3}{2}\frac{GM_\oplus}{c^2 r^3}\,\mathbf{x}\times\mathbf{v}, \tag{117}$$

where the following approximation has been used:

$$V_\oplus \approx \frac{GM_\oplus}{r} \quad\Rightarrow\quad \nabla V_\oplus = -\frac{GM_\oplus}{r^3}\,\mathbf{x}. \tag{118}$$

The geodetic precession is independent of $\mathbf{m}$, therefore it is not a gravitomagnetic effect.

The second term on the right hand side of equation (116) is called Lense-Thirring (or gravitomagnetic) precession and expresses the dragging of the surrounding space-time by a rotating mass. It can be written as

$$\mathbf{\Omega}_{LTP} = -\frac{GM_\oplus}{c^2 r^3}\left(\mathbf{J}_\oplus - \frac{3\,\mathbf{J}_\oplus\cdot\mathbf{x}}{r^2}\,\mathbf{x}\right), \tag{119}$$

because, using equation (112),

$$\mathbf{m}_\oplus = -\frac{GM_\oplus}{c^3 r^3}\begin{pmatrix} -J_\oplus x_2 \\ +J_\oplus x_1 \\ 0 \end{pmatrix} \Rightarrow \nabla\times\mathbf{m}_\oplus = \operatorname{rot}\mathbf{m}_\oplus = \frac{2GM_\oplus}{c^3 r^3}\left(\mathbf{J}_\oplus - \frac{3\,\mathbf{J}_\oplus\cdot\mathbf{x}}{r^2}\,\mathbf{x}\right). \tag{120}$$

In practice, the gravitomagnetic field of several other celestial bodies, e.g. Sun, may become significant. In case of an Earth orbiting satellite, the geodetic precession due to the Sun's influence will be superior to the same effect caused by the Earth. Furthermore, the effect of geodetic precession is by a few orders of magnitude larger than the effect of Lense-Thirring precession. As mentioned before, there exist minor additional effects, e.g., the so-called Thomas precession in case of non-zero local acceleration, i.e. if the observer is not in a free fall regime.

To incorporate relativistic precessional effects into the classical equation of motion of an Earth orbiting satellite

$$\ddot{\mathbf{r}} + \frac{GM_\oplus}{r^3}\,\mathbf{r} = \mathbf{f} \tag{121}$$

one would have to add a force term $\mathbf{f}_{\mathbf{\Omega}} = 2\mathbf{\Omega}\times\dot{\mathbf{r}}$ to the disturbing force $\mathbf{f}$. Details on spin-precessional effects are provided by Gill et al. [209].

29.14 Proper time and gravitational time delay

Aiming at the geodetic use of atomic clocks we apply the formula for the general proper time element, based on equations (17) and (49)

$$\mathrm{d}\tau^2 = -\frac{1}{c^2}\,g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j. \tag{122}$$

Explicitly, neglecting higher order terms but allowing for Coriolis force,

$$\mathrm{d}\tau^2 = -\left(g_{00} + \frac{v^2}{c^2}\right)\mathrm{d}t^2 - \frac{2}{c}\,\mathrm{d}t\big(g_{01}\mathrm{d}x + g_{02}\mathrm{d}y + g_{03}\mathrm{d}z\big) \tag{123}$$

with $v^2\mathrm{d}t^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

For clocks at rest at Earth's surface we have $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z = 0$ and therefore

$$\mathrm{d}\tau^2 = -g_{00}\mathrm{d}t^2 \quad\text{with}\quad g_{00} = -1 + 2\left(\frac{W}{c^2}\right), \tag{124}$$

because Earth's rotation transmits to the clock. Thus, the clock is subject to the centrifugal potential Φ, too.

For clocks in motion aboard planes or satellites we have $v \neq 0$ but $v \ll c$ and $g_{0i} = 0$, because now Earth's rotation has no direct effect on the clock. Of course, there exist indirect effects, but approximatively

$$\mathrm{d}\tau^2 = -\left(g_{00} + \frac{v^2}{c^2}\right)\mathrm{d}t^2 \quad\text{with}\quad g_{00} = -1 + 2\left(\frac{V}{c^2}\right). \tag{125}$$

Most geodetic applications will require differential clock measurements. Considering different earthbound positions, denoted by a subindex k, one gets

$$\mathrm{d}\tau_k = \sqrt{1 - \frac{2W_k}{c^2}}\,\mathrm{d}t. \tag{126}$$

The definition for coordinate time t provides $\mathrm{d}t_k = \mathrm{d}t$. Comparing two different positions/clock readings, yields

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} = \sqrt{\frac{1 - \frac{2W_2}{c^2}}{1 - \frac{2W_1}{c^2}}}\,. \tag{127}$$

Binomial series expansion and neglecting of higher order terms results in

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} \approx \left(1 - \frac{W_2}{c^2}\right)\left(1 + \frac{W_1}{c^2}\right) = 1 + \frac{W_1}{c^2} - \frac{W_2}{c^2} - \frac{W_1 W_2}{c^4} \approx 1 + \frac{W_1}{c^2} - \frac{W_2}{c^2}\,. \tag{128}$$

In the following we will use

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} = 1 + \frac{W_1 - W_2}{c^2} =: 1 - \frac{\Delta W}{c^2}\,. \tag{129}$$

(Moritz/Hofmann-Wellenhof [369]) introduce T_k as the periods of the individual clocks' atomic vibrations in their respective proper times, such that $\mathrm{d}\tau_2/\mathrm{d}\tau_1 = T_2/T_1$. The same periods with respect to coordinate time are all equal and denoted by $T := \mathrm{d}t_k = \mathrm{d}t$. Again making use of binomial series expansion, we find

$$\begin{aligned}
\frac{\Delta T}{T} := \frac{T_2}{T} - \frac{T_1}{T} = \frac{\mathrm{d}\tau_2}{\mathrm{d}t} - \frac{\mathrm{d}\tau_1}{\mathrm{d}t} &= \sqrt{1 - \frac{2W_2}{c^2}} - \sqrt{1 - \frac{2W_1}{c^2}} \\
&\approx \left(1 - \frac{W_2}{c^2}\right) - \left(1 - \frac{W_1}{c^2}\right) \\
&= \frac{W_1}{c^2} - \frac{W_2}{c^2} = -\frac{\Delta W}{c^2} = \frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} - 1\,.
\end{aligned} \tag{130}$$

In order to get a corresponding expression in terms of (proper) frequency with

$$f = \frac{1}{T} \quad \Rightarrow \quad \mathrm{d}f = -\frac{1}{T^2}\,\mathrm{d}T \quad \Leftrightarrow \quad T\mathrm{d}f = -\frac{1}{T}\,\mathrm{d}T \quad \Leftrightarrow \quad \frac{\mathrm{d}f}{f} = -\frac{\mathrm{d}T}{T}\,, \tag{131}$$

one just has to replace differentials by differences:

$$\frac{\Delta f}{f_1} = -\frac{\Delta T}{T_1} = \frac{\Delta W}{c^2} \quad \Leftrightarrow \quad \frac{f_2 - f_1}{f_1} = \frac{W_2 - W_1}{c^2}\,. \tag{132}$$

This simple but very important relation, representing a frequency shift due to gravitation (gravitational time delay) implies that, in principle, classical geodetic leveling techniques are replaceable by frequency measurements. This frequency shift may be superimposed by other effects, e.g., the Doppler effect that we have discussed before. Practically, the gravitational frequency shift can be detected in various ways, either directly or indirectly. Within the more practical part of this work we will provide additional remarks on the sign of the frequency shift, which might be helpful for comparison with other publications, cf. § 41.

29.15 Superposition and magnitude of individual relativistic effects

The use of accurate time or frequency measurements for the determination of geopotential differences has already been suggested decades ago, as mentioned in the introductory paragraphs. But only today or at least in the near future the necessary instrumentarium seems to be in reach in order to achieve an accuracy of $\Delta f/f = 10^{-18}$, which is actually required to determine geopotential differences that relate to height differences on the demandable *cm*-level.

In case of a moving clock, remember equation (125), we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{2V}{c^2} - \frac{v^2}{c^2}} \approx 1 - \frac{V}{c^2} - \frac{v^2}{2c^2}\,, \tag{133}$$

once more applying series expansion and neglecting terms of higher order.

As with the gravitomagnetic effects, any contributions stemming from different sources can be superimposed. Depending on the reference for the relative velocity v and the actual source of a gravitational potential, we may sort the significance of terms based upon its magnitude.

Considering the most important influences due to the Earth and the Sun only, one will find a

- term of order $O(10^{-8})$ due to Sun's gravitational potential, because at Earth's center of mass $V^{\odot}_{\text{Earth's cm}}/c^2 = 9.9 \cdot 10^{-9}$,
- term of order $O(10^{-9})$ due to Earth's gravitational potential, because at Earth's surface $V^{\oplus}_{\text{Earth's surface}}/c^2 = 7.0 \cdot 10^{-10}$,
- term of order $O(10^{-9})$ due to Earth's orbital motion about the Sun, because for an earthbound clock $(v^{\text{orb}}_{\oplus}/c)^2/2 = 4.9 \cdot 10^{-9}$,
- term of order $O(10^{-12})$ due to Earth's rotation, because for a clock located at Earth's equator $(v^{\odot}_{\oplus}/c)^2/2 = 1.2 \cdot 10^{-12}$,

if the following approximative formulas and nominal values are being used:

$$\begin{aligned}
v^{\text{rot}}_{\oplus} &= \omega_{\oplus} a_{\oplus} \sin\theta_A && \text{with} \quad \omega_{\oplus} = \frac{2\pi}{86164\,\text{s}} = 7.292123517 \cdot 10^{-5}\,\text{s}^{-1}, \\
& && \phantom{\text{with}} \quad a_{\oplus} = 6378.136\,\text{km}\,, \\
v^{\odot}_{\oplus} &= \frac{2\pi d^{\odot}_{\oplus}}{P^{\odot}_{\oplus}} && \text{with} \quad d^{\odot}_{\oplus} = 1\,\text{AU} = 149.598 \cdot 10^6\,\text{km}\,, \\
& && \phantom{\text{with}} \quad P^{\odot}_{\oplus} = 365.25\,\text{d} \cdot 86400\,\text{s}\,\text{d}^{-1} = 31557600\,\text{s}\,, \\
V^{\oplus}_{\text{Earth's surface}} &= \frac{GM_{\oplus}}{R_{\oplus}} && \text{with} \quad GM_{\oplus} = \mu_{\oplus} = 398600.4415\,\text{km}^3\text{s}^{-2}, \\
& && \phantom{\text{with}} \quad R_{\oplus} \approx a_{\oplus}\,, \\
V^{\odot}_{\text{Earth's cm}} &= \frac{GM_{\odot}}{d^{\odot}_{\oplus}} && \text{with} \quad GM_{\odot} = \mu_{\odot} = 132712440018\,\text{km}^3\text{s}^{-2}.
\end{aligned} \tag{134}$$

In principle, one had to use $W^{\oplus}_{\text{Earth's surface}}$ instead of $V^{\oplus}_{\text{Earth's surface}}$. Here we neglected the centrifugal potential, because its value $\Phi^{\oplus}_{\text{Earth's surface}}$ is two orders of magnitude smaller. Even in the most significant (extremal) case of an equatorial clock site we have

$$\Phi^{\oplus}_{\text{Earth's surface}} = \tfrac{1}{2}\,\omega^2_{\oplus} a^2_{\oplus} = 0.1\,\text{km}^2\text{s}^{-2} \ll V^{\oplus}_{\text{Earth's surface}} = 62.5\,\text{km}^2\text{s}^{-2}. \tag{135}$$

In order to perform time synchronization between portable atomic clocks or to compare clock readings between remote clock sites, one could relate accumulated infinitesimal proper time intervals with respect to a common coordinate time t, based on equation (133), i.e.,

$$t = \int \left(1 + \frac{V}{c^2} + \frac{v^2}{2c^2}\right) \mathrm{d}\tau, \tag{136}$$

resulting from just another series expansion.

As we have seen, relativistic effects directly affect geodetic measurements, either by deflecting the signal's path or by shifting the signal's frequency. The derivation of a final formula for the gravitational time delay is based on the line element (111) with neglect of all gravitomagnetic effects and other higher order terms. The derivation can be simplified if we choose the same special cartesian coordinate system that we already used in the discussion of signal bending, see the remarks following equation (92). Thus, the signal propagates parallel to the y-axis, i.e. $\mathrm{d}x, \mathrm{d}z = 0$, and there remains

$$\mathrm{d}s^2 = g_{00} c^2 \mathrm{d}t^2 + g_{22} \mathrm{d}y^2 = 0\,, \tag{137}$$

where

$$g_{00} = -1 + 2\,\frac{V}{c^2}\,, \quad g_{22} = 1 + 2\,\frac{V}{c^2} \quad \text{with} \quad V \approx \frac{GM}{r}\,. \tag{138}$$

The signal passes the attracting body of mass M at a minimal distance D, such that $r = \sqrt{D^2 + y^2}$. Solving equation (137) for t and considering the time interval between two positions A and B along the path yields

$$\Delta t_{AB} := t_B - t_A = \frac{1}{c} \int_A^B \sqrt{-\frac{g_{22}}{g_{00}}}\,\mathrm{d}y \approx \frac{1}{c} \int_A^B \left(1 + \frac{2GM}{c^2\sqrt{D^2 + y^2}}\right) \mathrm{d}y \tag{139}$$

and finally, setting $s_{AB} := y_B - y_A$,

$$\Delta t_{AB} = \frac{s_{AB}}{c} + \delta t_{AB} \quad \text{with} \quad \delta t_{AB} := \frac{2GM}{c^3} \ln\left(\frac{y_B + \sqrt{D^2 + y_B^2}}{y_A + \sqrt{D^2 + y_A^2}}\right). \tag{140}$$

The total signal travel time between two points A and B exceeds the classical travel time s_{AB}/c by a relativistically induced gravitational time delay δt_{AB}. The changing path length due to bending is of higher order and thus can be ignored in this approximative derivation.

Of course, this gravitational time dilation has to be accounted for twice in reflector measurements, e.g., within radar echo experiments to determine planetary distances or in case of lunar laser ranging. In addition, proper time corrections have to be applied, as discussed before.

Combination of the general Doppler effect (55) and gravitational time delay (132) results in

$$\frac{f_E - f_R}{f_E} = \frac{1}{c}\, v_r + \frac{1}{c^2}\left(v_r^2 - \tfrac{1}{2}v^2 + W_E - W_R\right) + O(c^{-3})\,, \tag{141}$$

where the first term on the right hand side represents the classical Doppler effect. Lower indices E and R denote emission and reception, respectively. The emitted frequency will change into a received frequency, depending on the relative velocity v between the sender and the receiver, where v_r is the radial component of v. Additionally, the total frequency shift depends on the difference in the gravity potential values of both sites. The smallness of all relativistic effects (excluding the longitudinal Doppler effect) allows its linear superposition.

30 Essential expressions for relativistic geodesy

30.1 Specific relations between coordinate time and proper time

Relating the individual readings of distributed atomic clocks to an uniformly valid coordinate time t comprises a series of time scale transformations (Soffel [511]). Each single atomic clock gives an individual proper time τ_k. Theoretically, atomic time may differ from proper time because we are not using ideal free falling clocks. Kopejkin et al. [299] pick up the issue of a photon clock. To a first approximation, we will assume earthbound individual clocks that are all spatially fixed with respect to each other. In practice, there exist tectonic motions, tidal effects etc., of course. Even disregarding any mutual movements, clock readings will differ not only due to systematic or stochastic errors, but because of our known simple relation (129).

Nowadays, the influence of different clock site heights, i.e. different potential values W_k, is being accounted for by relating all individual clock readings to a common equipotential surface, e.g. the traditional geoid. From $\Delta W \approx g\Delta H^o$ we can infer that a difference in (orthometric) height of 1 *cm* leads to an effect on the order of $O(10^{-18})$ on the ratio $\mathrm{d}\tau_2/\mathrm{d}\tau_1$, because $g\Delta H^o/c^2 \approx 9.81 \cdot 0.01/299792458^2 = 1.09 \cdot 10^{-18}$. Upcoming optical clocks will probably be sensitive enough to this accuracy level. The reduction of all individual measurements requires a priori knowledge of the potential values W_k and W_0, the latter being the geoidal geopotential value (Burša et al. [78], [79]). Remaining differences are then being averaged out by a standardized procedure (McCarthy/Seidelmann [351]) to form an uniformly valid international atomic time scale (TAI). Subsequent transformations via terrestrial dynamical time (TDT) and barycentric dynamical time (TDB) finally result in a (barycentric) coordinate time t (Seidelmann [496], McCarthy/Seidelmann [351]).

In relativistic geodesy we want to solve the inverse problem. One simply regards the relation between τ and t as an observation equation that allows us to solve for a variety of unknowns. To take full advantage of all the different effects that influence atomic clock readings, we can supplement the right hand side of a suitably chosen transformation equation $t - \tau$ by a consistent collection of individual terms that we know are significant to our actual instrumentation. Depending on the final quantities we are aiming at, the observation equation might be expressed in various equivalent forms. For instance, one may focus on earthbound unknowns, e.g. observation site positioning, and therefore emphasize the dependency on the Earth centered Earth fixed (ECEF) position of the atomic clock $\mathbf{r}_A^{\mathrm{ECEF}}$ itself (Moyer [370]). As a remark, today's precise measurement technology seems to be sensitive enough for the inclusion of at least the major planets Jupiter and Saturn. Using a more intuitive notation, one could finally write

$$t-\tau = \Delta T_A + \frac{1}{c^2}\left(2\sqrt{\mu_\odot a_\oplus^\odot}\, e_\oplus^\odot \sin E_\oplus^\odot + \frac{\mu_J}{v_J^\odot}\, e_J^\odot \sin E_J^\odot + \frac{\mu_S}{v_S^\odot}\, e_S^\odot \sin E_S^\odot + \dot{\mathbf{r}}_B^\odot \cdot \mathbf{r}_\oplus^B + \dot{\mathbf{r}}_\odot^C \cdot \mathbf{r}_A^\oplus + \dot{\mathbf{r}}_B^\odot \cdot \mathbf{r}_A^\oplus + \dot{\mathbf{r}}_\oplus^B \cdot \mathbf{r}_A^\oplus + \dot{\mathbf{r}}_\odot^C \cdot \mathbf{r}_B^\odot\right), \tag{142}$$

where A refers to the atomic clock, J, S to the planets Jupiter and Saturn, $\oplus$ to the Earth, $\odot$ to the Sun, B to the Earth-Moon barycenter, and C to the solar system barycenter. In case of both, upper and lower indices at the same letter, the upper one indicates the reference. For example, $\dot{\mathbf{r}}_B^\odot$ denotes the velocity vector of the Earth-Moon barycenter with respect to the Sun. The components of all vectors relate to the solar system barycentric space-time frame of reference. The remaining symbols are familiar from celestial mechanics: e (numerical eccentricity), a (semi-major axis), E (eccentric anomaly), v (velocity), and μ (gravitational parameter, $\mu = GM$). The first term on the right hand side collects a few terms that arise, if we formally integrate equation (133), i.e. (136). Integration from t_0 to t, among others, yields a term $t_0 - \tau_0$ as well as periodic terms at t_0, all of which are absorbed by ΔT_A. It will be nearly a constant for all earthbound atomic clocks that contribute to the atomic time scale. Its value is essentially the same as the nominal difference between TAI and TDT ($32.184\,s$).

We will now take a brief look at the individual bracket terms of equation (142). Applying appropriate astronomical planetary data (e.g. Allen [10], Cox [108], de Pater/Lissauer [131]) we find that the amplitudes (together with the factor $1/c^2$) of the first three (sine) terms are $1.658 \cdot 10^{-3}s$, $5.21 \cdot 10^{-6}s$, and $2.45 \cdot 10^{-6}s$, respectively. The remaining five terms, each expressed in form of a scalar product, can be characterized in the frequency domain by looking at the involved celestial bodies. Although each product comprises different motions, there will be dominant effects that govern the resulting frequency and amplitude of the term's signal. The first term (in the second row of equation (142)) is a monthly term, because Earth's movement around the Earth-Moon barycenter is involved and this effect is superior to a change of that barycenter with respect to the Sun. The second term is a daily one, thanks to the dominant position change of the clock simply due to Earth's rotation. Likewise, the third term comprises a daily proportion but at the same time the annual effect of the Earth-Moon barycenter motion around the Sun is significant. The fourth term shows a monthly and daily component again. The last term is mainly driven by a notable change in the Sun's position with respect to the solar system barycenter due to the gravitational attraction by the major planets Jupiter and Saturn. Consequently, this fifth term is synodic in nature.

Alternatively, one may be interested in space-related unknowns, e.g. planetary ephemeris, and therefore highlight genuine astronomical terms (Moyer [371]). One can rewrite each scalar product term in equation (142) by making use of well-known astronomical relations. The final result reads

$$
\begin{aligned}
t - \tau = \Delta T_A + \frac{1}{c^2}\Bigg(& 2\sqrt{\mu_\odot a_\oplus^\odot}\, e_\oplus^\odot \sin E_\oplus^\odot + \frac{\mu_J}{v_J^\odot}\, e_J^\odot \sin E_J^\odot + \frac{\mu_S}{v_S^\odot}\, e_S^\odot \sin E_S^\odot + \\
&+ \frac{v^B_{\odot_\mathrm{circ}} a_M^\oplus}{1+\mu} \sin D_{\odot M}^\oplus + \frac{v_M^\oplus (1+\cos\varepsilon)\,\xi_A^\oplus}{2(1+\mu)} \sin\big(UT1 + \lambda_A^\oplus - D_{\odot M}^\oplus\big) + \\
&+ \frac{(1+\cos\varepsilon)\,\xi_A^\oplus}{2\mu_C}\big(\Phi_J + \Phi_S\big) + \frac{a_\odot^B}{\mu_C}\big(\Psi_J + \Psi_S\big) - v^B_{\odot_\mathrm{circ}} \sin\varepsilon\, \eta_A^\oplus \cos L_\odot^B + \\
&+ \tfrac{1}{2} v^B_{\odot_\mathrm{circ}} (1+\cos\varepsilon)\,\xi_A^\oplus \Big(\big(1 - \tfrac{1}{2} e_\oplus^{\odot 2}\big) \sin\big(UT1 + \lambda_A^\oplus\big) + e_\oplus^\odot \sin\big(UT1 + \lambda_A^\oplus - M_\odot^C\big) + \\
&\qquad + \tfrac{9}{8} e_\oplus^{\odot 2} \sin\big(UT1 + \lambda_A^\oplus - 2M_\odot^C\big) - \tfrac{1}{8} e_\oplus^{\odot 2} \sin\big(UT1 + \lambda_A^\oplus + 2M_\odot^C\big)\Big) - \\
&- \tfrac{1}{2} v^B_{\odot_\mathrm{circ}} (1-\cos\varepsilon)\,\xi_A^\oplus \Big(\sin\big(UT1 + \lambda_A^\oplus + 2L_\odot^B\big) + e_\oplus^\odot \sin\big(UT1 + \lambda_A^\oplus + 2L_\odot^B + M_\odot^C\big)\Big)\Bigg)
\end{aligned}
\tag{143}
$$

with

$$
\begin{aligned}
\mu &= \mu_\oplus / \mu_M\,, \\
\Phi_J &= \mu_J v_J^\odot \sin\big(UT1 + \lambda_A^\oplus + L_\odot^B - L_J^\odot\big), \\
\Phi_S &= \mu_S v_S^\odot \sin\big(UT1 + \lambda_A^\oplus + L_\odot^B - L_S^\odot\big), \\
\Psi_J &= \mu_J v_J^\odot \sin\big(L_\odot^B - L_J^\odot\big), \\
\Psi_S &= \mu_S v_S^\odot \sin\big(L_\odot^B - L_S^\odot\big),
\end{aligned}
\tag{144}
$$

where ε is the usual symbol for the obliquity of the ecliptic, $\xi_A^\oplus$ and $\eta_A^\oplus$ are the orthogonal distances of the earthbound atomic clock from the rotational axis and the equatorial plane, respectively. Both values can easily be calculated from given (geographical) coordinates of the clock site. The additional subindex M represents the Moon, and $_\mathrm{circ}$ indicates that a corresponding approximative value for an assumed circular orbit is sufficient. The angular arguments are: elongation D (angle between two celestial bodies as seen from a third one), ecliptical longitude L (referred to the mean equinox), mean anomaly M, and geographical longitude λ. The independent time variable is provided by the argument $UT1$ (Schödlbauer [485]). Usually, variable astronomical quantities, especially the angles D, L, and M, are calculated by means of series expansions, i.e., polynomials in time (Seidelmann [496]). Alternatively, one applies orbital integration techniques for the computation of the ephemeris. Radar observations, besides other observation techniques, are being used to create the most accurate ephemeris files, e.g., the DE-files by JPL. In principle, pure time observations of earthbound clocks should be applicable to at least partially determine the ephemeris of major solar system bodies. This provides a complete independent method, without looking into the sky at all. Of course, this method is only a theoretical one, because its accuracy will probably never reach the levels of established direct methods. Rather in relativistic geodesy, we apply the underlying formulas of time and/or frequency comparisons for the purpose of positioning and large distance height transfer, focussing on the occurrence of $\mathbf{r}_A^{\oplus}$ or $\xi_A^{\oplus}$, $\eta_A^{\oplus}$, and $\lambda_A^{\oplus}$ within the observation equation. These are our primary unknowns. Performing transoceanic height transfers on a global scale could be labeled as intercontinental leveling.

Equations (142) and (143) are equivalent, their usability solely depends on the actual application. Moyer [370], [371], in order to derive his equation for $t - \tau$, made use of several simplifying assumptions. These were justified by foreseeable applications and available contemporary instrumentation, that this author kept in mind.

Since the early 1980's significant progress has been achieved regarding measurement technology. Therefore, one should revise higher order terms that were neglected by Moyer and other authors in the past. This statement certainly holds true not only for the derivation of astronomical formulas, but also for the consideration of higher order relativistic terms within the fundamental metric tensors.

The gravitational potential and the centrifugal potential of the Earth give raise to relativistic (correction) terms of order (Kopejkin et al. [299])

$$\frac{1}{c^2}\frac{GM_\oplus}{R_\oplus} \approx 0.7 \cdot 10^{-9}, \quad \text{and} \quad \frac{1}{c^2}\,\omega_\oplus^2 R_\oplus^2 \approx 2.4 \cdot 10^{-12}, \tag{145}$$

respectively. These general relativistic terms are superimposed by special relativistic terms (whose order depends on the actual relative velocity), remember equations (133) or (141). The above stated orders of magnitude are well within the range of accuracy of current instrumentation. Conventional ballistic gravimeters are sensitive to the $1\mu gal$-level, where $1gal = 10^{-2}m/s^2$. Superconducting gravimeters exceed this level approximately by a factor of 1000. Therefore, the achievable relative accuracy is about $10^{-3}\mu gal/g \approx 10^{-12}$ (with $g \approx 981 \cdot 10^6 \mu gal$) such that relativistic gravimetry no longer remains a pure theoretical idea. The same holds true for relativistic gradiometry. Today gradients of g (dimension: acceleration/distance) can be measured with mE-level precision, where $1E = 10^{-9}s^{-2}$, using superconducting instruments again. As mentioned in the introductory part, novel quantum engineering techniques make use of atom interferometry which eventually led to gravity gradiometers that are sensitive enough to be used in precision gravity experiments (McGuirk et al. [352], Fixler et al. [187]).

30.2 Problem-dependent fixing of the tensors

For earthbound relativistic approaches the solution of Einstein's field equations in the vicinity of Earth's center of mass becomes especially important. It requires a specific definition of the corresponding energy-momentum tensor T_{ij} or T^{ij} of Earth's matter, including various sources for a gravitational field. Specifically, T^{ij} also incorporates anisotropic stresses (Kopejkin et al. [299]), expressed by a stress tensor π^{ij}. More realistic equations additionally account for other parameters too, like viscosity or elasticity, because in relativity several sources of a gravitational field exist, e.g. energy density, pressure, and stresses. Consequently, different kinds of energy will contribute, like kinetic energy, gravitational potential energy, energy from deformations, etc. (Soffel et al. [512]). Any additional parameters could be determined from geodetic measurements.

Certain solution processes involve post-Newtonian iterations, where physical quantities like the four-velocity $u^i = \mathrm{d}x^i/c\mathrm{d}\tau$ or tensors g_{ij} and π^{ij} are expanded in powers of c^{-1}. These expansions are inserted into the field equations.

Soffel et al. [512] provide a detailed explanatory supplement on a set of recent IAU resolutions regarding the consistent definition of metric tensors, gravitational potentials and reference systems. The self-consistency of the relativistic framework is of uppermost importance, because the significance of various relativistic effects changes according to the underlying reference system. This article also stresses that the basic formalism should not simply be expressed in terms of small relativistic corrections to Newtonian theory. As we have seen before, in general relativity there do not exist globally valid spatial inertial coordinates. Rather, space-time coordinates in general no longer have a direct physical meaning and therefore all observables should be constructed as coordinate-independent quantities. Relativistic modeling of any observation requires a consistent relativistic four-dimensional reference system as described by x^i and materialized by a corresponding reference frame.

To fix a particular reference system one has to specify the entries of the metric tensor g_{ij}. Being the starting point, the metric tensor allows for the subsequent derivation of equations of motion (translational as well as rotational), equations for the propagation of signals, and for the modeling of the observation process. Depending on the suitable choice of the reference system, the resulting models and parameters become simple and physically adequate, respectively.

In order to determine the equations of motion for a given model of matter one has to find a suitable formulation of the corresponding gravitational field. This requires us to specify a gauge condition on the metric tensor, i.e. on the field variables, in order to reduce the existing redundant degrees of freedom in those variables. The main reason for the existence of (four) arbitrary functions in the general solution is the fact that we have intentionally chosen the use of covariant field equations which are therefore valid in any coordinates. Imposing gauge conditions significantly narrows down the arbitrariness. By choosing a certain gauge condition we can simplify the field equations to our special needs. Afterwards we have to solve these reduced equations for the given metric tensor (plus possibly other fields, e.g. scalar ones), and finally derive the equations of motion that are consistent with our solutions of the field equations (Kopejkin et al. [299]). For details on the gauge theory see for instance Hehl [238].

30.3 BK-approach vs DSX-approach

Several well-established reference systems are involved in astronomical and geodetic observations. A detailed discussion on the notation and relations between the (solar system) barycentric and geocentric celestial reference systems (BCRS, GCRS) can be found in Soffel et al. [512]. Basically, two competitive approaches were elaborated in the past. The first one by Brumberg/Kopejkin [73] (BK-approach) is based on the asymptotic matching technique for the gravitational field potentials, where matching refers to the BCRS and GCRS metric tensor components and relies on the existence of several small parameters: e.g., the slowness of involved motions of the bodies can be characterized by a velocity parameter v as the maximum of typical internal values of $\{|T_{0j}|/|T_{00}|, |T_{ij,0}|/|T_{ij,k}|\}$, the weakness of the gravitational fields outside and inside the bodies can be characterized by a parameter ϵ as the maximum over the entire source of $\bar{h}^{00}$ with $\bar{h}^{ij} = -\sqrt{-g}\, g^{ij} + \eta^{ij}$, and small stresses can be characterized by a parameter S as the maximum over the entire source of $|T_{ij}|/|T_{00}|$ (Thorne [548]). Additionally, the structure of the bodies, Earth oblateness etc. is traditionally characterized by scalar spherical harmonic coefficients c_{lm} and s_{lm}.

Stating that the matching approach may have some severe drawbacks (Kopejkin et al. [299]), Damour et al. [115], [116], [117], [118] developed an alternative approach (DSX-approach) comprising a linearization of the Einstein field equations by means of an exponential parametrization of g_{ij}. The metric tensor to be found as the solution of the Einstein field equations simultaneously describes both the reference system and the gravitational field. As a consequence, in order to fix a solution one may impose arbitrary (coordinate) conditions on the metric tensor components. Among others the above mentioned approaches also differ in the treatment of this gauge degree of freedom.

Whereas in the DSX-approach the more flexible algebraic conditions

$$g_{00}g_{ij} = -\delta_{ij} + O\big(c^{-4}\big) \tag{146}$$

are being imposed, more restrictive approaches like the BK-approach impose a certain differential coordinate condition, e.g., the harmonic gauge condition

$$g^{ij}\Gamma^k_{ij} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x^i}\big(\sqrt{-g}\, g^{ij}\big) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^i}\, \bar{h}^{ij} = 0. \tag{147}$$

30.4 Celestial reference system connected to the (solar-system) barycenter

The overall idea in using several reference systems is to construct a suitable local reference system for each subsystem that is involved in the observation process. Regarding each local system, the influence of external matter is given by tidal potentials only, expanded in a power series with respect to local spatial coordinates. The occurrence of (fictitious) inertial forces, represented by linear terms in the series expansion, may be eliminated by a suitable choice of the local coordinates' origin.

The BCRS denoted by time and space coordinates $(t, \mathbf{x})$, neglecting non-solar system matter (model of an isolated solar system, i.e. without any tidal influence of other matter in the Milky Way or beyond on this system), is considered to be inertial and serves for astronomical applications, e.g. star catalogues, solar system ephemerides, interplanetary navigation. The GCRS denoted by time and space coordinates $(T, \mathbf{X})$, is called quasi-inertial due to the accelerated geocenter. It is being used for applications related to the vicinity of the Earth and serves as the basis for derived concepts like the international terrestrial reference system (ITRS). BCRS and GCRS are related by a generalized Lorentz transformation containing acceleration terms and gravitational potentials. Regarding the gravitational potentials it is assumed that they vanish far from the system.

There still exists a certain degree of freedom in the definition of the BCRS's orientation of the spatial axes. A natural choice is provided by the international celestial reference system (ICRS) as materialized by the international celestial reference frame (ICRF) via the Hipparcos catalogue, or an improved version ICRF2 which is based on revised and additional positions, more (VLBI-) observations and better analysis methods. These realizations lead to a specific barycentric metric tensor following the recommended general form. For highly precise applications the positions of stars (or remote astronomical objects in general) should solely refer to the BCRS.

The metric tensor of the BCRS, where the time coordinate ct is related to $t = TCB$ (barycentric coordinate time), is defined as (Soffel et al. [512])

$$
\begin{aligned}
g_{00} &= -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + O\big(c^{-5}\big), \\
g_{0i} &= -\frac{4w^i}{c^3} + O\big(c^{-5}\big) \qquad\qquad i = 1,2,3\,, \\
g_{ij} &= \delta_{ij}\left(1 + \frac{2w}{c^2}\right) + O\big(c^{-4}\big) \qquad i,j = 1,2,3\,,
\end{aligned}
\tag{148}
$$

where the traditional Newtonian potential is relativistically generalized by a scalar potential $w(t, \mathbf{x})$ and a vector (gravitomagnetic) potential $w^i(t, \mathbf{x})$. Due to the w^2-term, the metric is obviously not flat.

As an aside, spherical fields are of great importance in (geo-)research and thus one should consider essential features of spherical vector and tensor structures, too. Consequently, the generalization of fundamental concepts in the theory of spherical fields, e.g., Legendre polynomials, radial basis functions, splines, or wavelets, had to be generalized to the vectorial and tensorial case. Remembering the uncertainty principle we know that space localization and momentum (frequency) localization are mutually exclusive. Some extremal functions in this sense are the Legendre kernels (no space localization, ideal frequency location), and the Dirac kernels (ideal space location, no frequency location). Freeden/Schreiner [199] show details on the transformation between those kernels and how to find an applicable compromise. Their approach comprises two separate transitions, from spherical harmonics to Dirac kernels, and from scalar theory to vector and tensor theory, respectively. By choosing a coordinate-free setup they are able to avoid any kind of singularity at the poles, which is often a problem in classical approaches.

Choosing the condition (147) the post-Newtonian Einstein field equations take the form

$$
\begin{aligned}
\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) w &= -4\pi G\sigma + O\big(c^{-4}\big), \\
\Delta w^i &= -4\pi G\sigma^i + O\big(c^{-4}\big) \qquad i = 1,2,3\,,
\end{aligned}
\tag{149}
$$

with the gravitational mass $\sigma(t, \mathbf{x})$ and the mass current density $\sigma^i(t, \mathbf{x})$, where the term 'mass currents' refers to moving or rotating masses. These quantities are related to the energy momentum tensor T^{ij} (Soffel et al. [512])

$$
\sigma = \frac{1}{c^2}\big(T^{00} + T^{kk}\big) + O\big(c^{-4}\big), \quad \sigma^i = \frac{1}{c}\, T^{0i} + O\big(c^{-2}\big),
\tag{150}
$$

which generalizes the density ρ of the Poisson equation (80) and represents the gravitational sources. This tensor does not show up explicitly in practical computations, because gravitational fields outside extended bodies are traditionally given in form of convergent series expansions in terms of multipole moments, i.e., potential coefficients. These are defined by σ and σ^i, which describe the body's interior in terms of distributed mass and mass currents. Its values are estimated from observations outside of the body. In this respect, we handle relativistic multipole moments just in the same way as conventional scalar spherical harmonic coefficients. The latter are Newtonian multipole moments, often depending explicitly on time because the gravitating body may oscillate, wobble and change its internal structure. Additionally, as in case of the Earth notable mass variations in the body's exterior may exist, e.g., varying ice coverage in Greenland or global atmospheric currents and air tides, which are then also reflected in the gravitational potential. For the ease of modeling one often assumes that each body consists of viscoelastic matter which admits continuous mass density distributions, anisotropic stresses, and internal velocity fields. Under certain assumptions the lower degree moments remain constant (Kopejkin et al. [299]).

Coming back to the earlier assumption of an asymptotically flat space-time, which acts as a boundary condition, we find for $t = \text{const.}$

$$
\lim_{|\mathbf{x}|\to\infty} g_{ij} = \eta_{ij} \quad \Rightarrow \quad \lim_{|\mathbf{x}|\to\infty} w(t, \mathbf{x}) = 0\,, \quad \lim_{|\mathbf{x}|\to\infty} w^i(t, \mathbf{x}) = 0\,,
\tag{151}
$$

and the recommended solutions in form of volume integrals over the whole three-space will read

$$
\begin{aligned}
w(t,\mathbf{x}) &= G\!\int \frac{\sigma(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x' + \frac{G}{2c^2}\frac{\partial^2}{\partial t^2}\!\int \sigma(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|\,\mathrm{d}^3x' + O\big(c^{-4}\big),\\
w^i(t,\mathbf{x}) &= G\!\int \frac{\sigma^i(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x' + O\big(c^{-2}\big).
\end{aligned} \tag{152}
$$

Kopejkin et al. [299] also highlight the physical difference of coordinates. In contrast to global coordinates, with local coordinates we have a metric tensor that diverges with growing coordinate distance from the body because its gravitational field must smoothly match with any tidal gravitational fields in some region between the bodies. There is no asymptotically flatness analogically with equation (151).

In geodetic applications restricted to the gravitational fields outside massive bodies any observable will be determined by the sum of all metric potentials. Therefore, an artificial split into various pieces is not recommended (Soffel et al. [512]). As for the Newtonian potential in traditional satellite geodesy, suitably defined potential coefficients based upon w and w^i could be determined from satellite data.

In case of an N-body system, one may simply use

$$
w(t,\mathbf{x}) = \sum_{A=1}^{N} w_A(t,\mathbf{x})\,, \quad w^i(t,\mathbf{x}) = \sum_{A=1}^{N} w^i_A(t,\mathbf{x})\,, \tag{153}
$$

where the lower index A denotes the mere contribution of body A. For an approximative solution one only retains the mass monopoles of the bodies and therefore, with $\mu_A = GM_A$,

$$
\begin{aligned}
w(t,\mathbf{x}) &= \sum_A \frac{\mu_A}{r_A}\left(1+\frac{1}{c^2}\left(2v_A^2 - \sum_{B\neq A}\frac{\mu_B}{r_{BA}} - \frac{\big(r_A^k v_A^k\big)^2}{2r_A^2} - \frac{r_A^k a_A^k}{2}\right)\right),\\
w^i(t,\mathbf{x}) &= \sum_A \frac{\mu_A}{r_A}\, v_A^i\,,
\end{aligned} \tag{154}
$$

where $\mathbf{v}_A = \dot{\mathbf{x}}_A$, $\mathbf{a}_A = \dot{\mathbf{v}}_A$ and $\mathbf{r}_{BA} = \mathbf{x}_B - \mathbf{x}_A$.

An isolated ideal fluid can be characterized by the invariant densities $\rho(t,\mathbf{x})$ or $\rho^*(t,\mathbf{x}) = \rho(t,\mathbf{x})\sqrt{-g}\,c\mathrm{d}t/\mathrm{d}s$, the specific energy density $\Pi(t,\mathbf{x})$, the isotropic pressure $p(t,\mathbf{x})$, and the matter's velocity $v^i(t,\mathbf{x})$. As a remark, to some extent in conflict with Einstein's purist views, Fock [192] already stressed the importance of additional initial or boundary conditions in order to identify unique solutions to the (partial differential) field equations that link the fundamental and energy-momentum tensors to each other. Eventually this violates the postulation of general covariant equations. Following Klioner/Soffel [288] the corresponding energy momentum tensor reads

$$
\begin{aligned}
T^{00} &= \rho^* c^2\Big(1+\frac{1}{c^2}\big(\tfrac{1}{2}\,v^2+\Pi-V\big)\Big) + O\big(c^{-2}\big),\\
T^{0i} &= \rho^* c v^i\Big(1+\frac{1}{c^2}\big(\tfrac{1}{2}\,v^2+\Pi-V\big)\Big) + \frac{1}{c}\,pv^i + O\big(c^{-3}\big),\\
T^{ij} &= \rho^* v^i v^j + \delta^{ij}p + O\big(c^{-2}\big),
\end{aligned} \tag{155}
$$

where $V(t,\mathbf{x})$ represents a Newtonian-like gravitational potential, i.e.,

$$
V(t,\mathbf{x}) = G\!\int \frac{\rho^*(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x'. \tag{156}
$$

The post-Newtonian potentials $w(t,\mathbf{x})$ and $w^i(t,\mathbf{x})$ are formally given by

$$
\begin{aligned}
w(t,\mathbf{x}) &= V(t,\mathbf{x}) + \frac{G}{c^2}\!\int\!\left(\frac{\rho^*(t,\mathbf{x}')\big(\frac{3}{2}\dot{x}'^2+\Pi(t,\mathbf{x}')-V(t,\mathbf{x}')\big)+3p(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right. +\\
&\qquad\left. +\frac{1}{2}\frac{\partial^2}{\partial t^2}\Big(\rho^*(t,\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|\Big)\right)\mathrm{d}^3x',\\
w^i(t,\mathbf{x}) &= G\!\int \frac{\rho^*(t,\mathbf{x}')\,v^i(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x'.
\end{aligned} \tag{157}
$$

The volume integrals in equations (156) and (157) can be split into local and external parts.

30.5 Celestial reference system connected to the geocenter

As with the BCRS, Soffel et al. [512] discuss corresponding IAU recommendations regarding the GCRS. Its metric tensor shall be in the same form as the barycentric one. Again introducing a scalar potential $W(T, \mathbf{X})$ and a vector (gravitomagnetic) potential $W^a(T, \mathbf{X})$ by analogy to equation (152), we thus have

$$\begin{aligned}
\mathcal{G}_{00} &= -1 + \frac{2W}{c^2} - \frac{2W^2}{c^4} + O(c^{-5}), \\
\mathcal{G}_{0a} &= -\frac{4W^a}{c^3} + O(c^{-5}) \qquad\qquad a = 1,2,3\,, \\
\mathcal{G}_{ab} &= \delta_{ab}\left(1 + \frac{2W}{c^2}\right) + O(c^{-4}) \qquad a,b = 1,2,3\,.
\end{aligned} \tag{158}$$

This time it is recommended to split the metric into a local (Earth-related) part (denoted by $_\oplus$) and an external part (denoted by $_\text{ext}$) that collects all remaining terms stemming from inertial and tidal forces (denoted by $_\text{inert}$ and $_\text{tidal}$):

$$\begin{aligned}
W &= W_\oplus + W_\text{ext} = W_\oplus + W_\text{inert} + W_\text{tidal}\,, \\
W^a &= W^a_\oplus + W^a_\text{ext} = W^a_\oplus + W^a_\text{inert} + W^a_\text{tidal}\,.
\end{aligned} \tag{159}$$

Similar to the classical series expansion of the Newtonian potential we can apply a power series multipole expansion to the potentials W and W^a in terms of $R := |\mathbf{X}|$.

Denoting by $\mathcal{T}^{ij}$ the energy momentum tensor in the local reference system, e.g. GCRS, we introduce, likewise to equations (150), the quantities

$$\Sigma = \frac{1}{c^2}\left(\mathcal{T}^{00} + \mathcal{T}^{kk}\right) + O(c^{-4}), \quad \Sigma^a = \frac{1}{c}\,\mathcal{T}^{0a} + O(c^{-2}). \tag{160}$$

Thus, the local internal gravitational potentials $W_\oplus$ and $W^a_\oplus$ are supposed to have the same functional form as its barycentric counterparts (152):

$$\begin{aligned}
W_\oplus(T, \mathbf{X}) &= G\int_\oplus \frac{\Sigma(T, \mathbf{X}')}{|\mathbf{X} - \mathbf{X}'|}\,\mathrm{d}^3X' + \frac{G}{2c^2}\frac{\partial^2}{\partial T^2}\int_\oplus \Sigma(T, \mathbf{X}')|\mathbf{X} - \mathbf{X}'|\,\mathrm{d}^3X' + O(c^{-4}), \\
W^a_\oplus(T, \mathbf{X}) &= G\int_\oplus \frac{\Sigma^a(T, \mathbf{X}')}{|\mathbf{X} - \mathbf{X}'|}\,\mathrm{d}^3X' + O(c^{-2}).
\end{aligned} \tag{161}$$

If the actual internal gravitational field shows any deviation from the assumed form, which would imply a violation of the strong equivalence principle, then one would have to supplement the first equation of (159) by an additional term (Klioner/Soffel [288]). Of course, any alternative explanations for such a deviation, e.g., systematic errors due to mismodelling or inappropriate series expansions in practical computation, must first be ruled out.

The scalar inertial potential W_inert arises due to the fact that the actual world line of the geocenter deviates from a geodesic in the external gravitational field. This deviation is caused by the coupling of higher order multipole moments of the Earth with external tidal gravitational fields, the latter being related to the space-time curvature tensor as we have seen in the previous subsection. The coupling terms introduce an ambiguity in the definition of the multipoles in local coordinates (Kopejkin et al. [299]). The question arises whether to include or exclude the contribution of these terms to the definition of the central body's multipole moments. In fact, its inclusion will lead to simpler final equations of motion.

The above mentioned deviation vanishes in case of neglected multipole moments, i.e. if one only allows for mass monopoles (e.g. spherical non-rotating Earth). Otherwise it is characterized by the approximative quantity (Soffel et al. [512])

$$Q_a = \delta_{ai}\left(\frac{\partial}{\partial x^i}\,w_\text{ext}(t, \mathbf{x}_\oplus) - a^i_\oplus\right), \tag{162}$$

such that the inertial force arising due to the accelerated motion of Earth's center of mass can be described by (Klioner/Voinov [286])

$$W_\text{inert} = Q_a X^a, \tag{163}$$

where $\mathbf{x}_\oplus$, $\mathbf{v}_\oplus = \dot{\mathbf{x}}_\oplus$ and $\mathbf{a}_\oplus = \dot{\mathbf{v}}_\oplus$ refer to the time dependent barycentric coordinate position, velocity and acceleration of the geocenter (origin of the GCRS), respectively. The external potentials with respect to BCRS, i.e. w_ext and w^i_ext, are given by equations (153) without the contribution of Earth, i.e. for $A \neq \oplus$.

The vectorial inertial potential W^a_{inert} accounts for the relativistic Coriolis force in case of a rotating geocentric reference system. Our specific GCRS is defined as kinematically non-rotating with respect to BCRS, as indicated by the δ_{ai} in equation (162), but it rotates with respect to a dynamically non-rotating general geocentric reference system. This effect can be described by a precessional vector

$$\mathbf{\Omega}_{\mathrm{inert}} = \mathbf{\Omega}_{\mathrm{GP}} + \mathbf{\Omega}_{\mathrm{LTP}} + \mathbf{\Omega}_{\mathrm{TP}}, \tag{164}$$

which we know already from equation (116), where we neglected the Thomas precession. Using the ε-tensor (Moritz/Hofmann-Wellenhof [369]), i.e. the fully antisymmetric Levi-Civita symbol (Levi-Civita pseudo tensor), we may express vector cross products via index notation, e.g. $c_i = \varepsilon_{ijk}\, a_j b_k$ is equivalent to $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. One finally gets (Soffel et al. [512])

$$W^a_{\mathrm{inert}} = -\tfrac{1}{4}\, c^2 \varepsilon_{abc}\, \Omega^b_{\mathrm{inert}} X^c. \tag{165}$$

A series expansion of $w_{\mathrm{ext}}(t, \mathbf{x})$

$$w_{\mathrm{ext}}(t, \mathbf{x}_\oplus + \mathbf{X}) = w_{\mathrm{ext}}(t, \mathbf{x}_\oplus) + \nabla w_{\mathrm{ext}}(t, \mathbf{x}_\oplus) \cdot \mathbf{X} + \cdots \tag{166}$$

relates to the Newtonian tidal potential

$$W^{\mathrm{Newton}}_{\mathrm{tidal}}(T, \mathbf{X}) = w_{\mathrm{ext}}(t, \mathbf{x}_\oplus + \mathbf{X}) - w_{\mathrm{ext}}(t, \mathbf{x}_\oplus) - \nabla w_{\mathrm{ext}}(t, \mathbf{x}_\oplus) \cdot \mathbf{X}. \tag{167}$$

Using index notation, the last term in equation (167) could alternatively be written as $-w_{\mathrm{ext},i}(t, \mathbf{x}_\oplus) X^i$, and a post-Newtonian expansion of the tidal potential in powers of $\mathbf{X}$ would read (Klioner/Voinov [286])

$$W_{\mathrm{tidal}}(T, \mathbf{X}) = \tfrac{1}{2}\, Q_{ij} X^i X^j + \tfrac{1}{6}\, Q_{ijk} X^i X^j X^k + \cdots \tag{168}$$

with (Kopejkin et al. [299])

$$Q_{ij} = w_{\mathrm{ext},ij}(t, \mathbf{x}_\oplus) + O\big(c^{-2}\big), \quad Q_{ijk} = w_{\mathrm{ext},ijk}(t, \mathbf{x}_\oplus) + O\big(c^{-2}\big), \quad \text{etc.} \tag{169}$$

30.6 Classical spherical harmonics and relativistic multipole moments

The moments (169) could be shortened by introducing a condensed notation (Blanchet/Damour [49]), e.g., by replacing sequences of spatial indices (i.e., with each individual index taking the values 1,2, and 3) either by a single upper-case latin letter index

$$ijk\cdots = i_1 i_2 i_3 \cdots i_l =: L, \quad i_1 i_2 i_3 \cdots i_{l-1} =: L-1, \quad \cdots, \tag{170}$$

or in a slightly different way (Thorne [548]) by

$$ijk\cdots = a_1 a_2 a_3 \cdots a_l =: A_l, \quad a_1 a_2 a_3 \cdots a_{l-1} =: A_{l-1}, \quad \cdots. \tag{171}$$

The tidal gravitational quadrupole and octupole moments as given in equation (169) without explicit relativistic terms are sufficient for ground-based geodetic applications.

In classical mechanics, based upon Newtonian potentials representing scalar fields, one usually applies the general form of scalar spherical harmonics

$$Y_{lm}(\theta, \phi) = \begin{cases} C_{lm}\, e^{im\phi} P_{lm}(\cos\theta) & \text{for} \quad m \geq 0 \\ (-1)^m Y^*_{l|m|} & \text{for} \quad m < 0 \end{cases}, \tag{172}$$

where indices are not to be confused with usual index notation for tensors, i.e. summation rules do not apply here. The $*$ denotes complex conjugation and $P_{lm}(\cos\theta)$ are the classical associated Legendre functions of degree l and order m. The scalar spherical harmonic coefficients are defined by

$$C_{lm} := (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \tag{173}$$

and under complex conjugation the scalar spherical harmonic coefficients transform as

$$Y^*_{lm} = (-1)^m Y_{l\,-m}. \tag{174}$$

The definitions (172) and (173) are adopted from the physicist Thorne [548], whereas in geodesy one uses a slightly different version, where basically the 4π factor is kept outside the whole Y_{lm} definition.

In physics, one has to deal with more complicated fields, namely vector fields, e.g. in electromagnetism, or tensor fields, e.g. in general relativity. Accordingly, more general spherical harmonic multipole expansions have to be applied. If non-linearity is added, the whole problem becomes even more complicated since the multipole components are coupled together by field equations. General relativity is just an example, where both difficulties come together: non-linearity and tensor fields due to the metric tensor. Several general relativistic multipole expansions do exist in literature, which is reviewed and consolidated in Thorne [548]. There is no unique procedure to find the gravitational field from a given metric as in the Newtonian theory of gravity. Consequently, one has to study different proposed relativistic multipole definitions and its relations (Quevedo [438]).

The solution of the classical Poisson equation (80) in the stationary case, i.e. $V = V(\mathbf{X})$, can either be written traditionally as ($R = |\mathbf{X}|$)

$$V(\mathbf{X}) = 4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Q_{lm}}{2l+1} \frac{Y_{lm}(\Theta,\Phi)}{R^{l+1}}\,, \tag{175}$$

temporarily denoting field point related quantities by upper-case letters and source point quantities by lower-case letters, with

$$Q_{lm} = \int Y^*_{lm}(\theta,\phi)\, r^l \rho(\mathbf{x})\, \mathrm{d}^3x\,, \tag{176}$$

or as (Damour/Iyer [114]), using the condensed index notation,

$$V(\mathbf{X}) = G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, Q_L\, \partial_L \left(\frac{1}{R}\right) \tag{177}$$

with $\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$, $\partial_i = \partial/\partial x^i$, and

$$Q_L = \int x^{\langle i_1} \cdots x^{i_l\rangle} \rho(\mathbf{x})\, \mathrm{d}^3x\,. \tag{178}$$

Angular brackets or a caret indicate symmetric trace-free (STF) tensors (Blanchet/Damour [49]), i.e. irreducible Cartesian tensors, whereas round brackets denote symmetrization only. Finally, square brackets indicate anti-symmetrization. As a remark, the STF part being equal to zero means isotropy.

Exemplarily, assuming a tensor symbol A we write (Klioner/Soffel [288]) for $l = 2$

$$\begin{aligned} A_{[ij]} &= \tfrac{1}{2}\big(A_{ij} - A_{ji}\big), \\ A_{(ij)} &= \tfrac{1}{2}\big(A_{ij} + A_{ji}\big), \\ \widehat{A}_{ij} = A_{\langle ij\rangle} &= A_{(ij)} - \tfrac{1}{3}\delta^{ij} A_{cc}\,. \end{aligned} \tag{179}$$

This notation can be extended to any value of l (Damour/Iyer [114]), e.g., $l = 3$ yields (Kopejkin et al. [299])

$$\widehat{A}_{ijk} = A_{\langle ijk\rangle} = A_{(ijk)} - \tfrac{1}{5}\big(\delta^{ij} A_{kcc} + \delta^{jk} A_{icc} + \delta^{ki} A_{jcc}\big). \tag{180}$$

If convenient, the condensed notation can be used, e.g., $\widehat{A}_{i_1 i_2\cdots i_l} = \widehat{A}_L$ or $A_{\langle i_1 i_2\cdots i_l\rangle} = A_{\langle L\rangle}$ and $A_{(i_1 i_2\cdots i_l)} = A_{(L)}$.

Regarding the relativistic time-dependent multipole expansion comprising tensorial fields, at least in case of linearized gravity the Cartesian multipole approach is algebraically more transparent than a potential formalism that was developed for vector fields.

Exploiting the concept of STF tensors one can express the spherical harmonics via

$$Y_{lm}(\theta,\phi) = \widehat{Y}^{lm}_L n^L, \tag{181}$$

where $\widehat{Y}^{lm}_L$ is a location-independent STF tensor as defined by Thorne [548] with $-l \le m \le +l$ and n^L is the tensorial product of l radial unit vectors. Those $\widehat{Y}^{lm}_L$ not only generate the $Y_{lm}(\theta,\phi)$ but may also be used to formally expand any other STF tensor $\widehat{F}_L$ into a series

$$\widehat{F}_L = \sum_{m=-l}^{l} F_{lm}\, \widehat{Y}^{lm}_L. \tag{182}$$

On a sphere centered on the coordinate center, we are now able to express a scalar function $f(\theta,\phi)$ by an infinite series in powers of the unit radial vector

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \widehat{F}_L n^L, \tag{183}$$

which is related to the traditional spherical harmonics representation

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} F_{lm} Y_{lm}(\theta,\phi) \tag{184}$$

via the coefficients F_{lm} that can be written as

$$F_{lm} = 4\pi \frac{l!}{(2l+1)!!}\, \widehat{F}_L\, \widehat{Y}_L^{lm*}. \tag{185}$$

The pointwise convergence of traditional spherical harmonics expansions can be controlled by the identity

$$\sum_{m=-l}^{l} \left|Y_{lm}(\theta,\phi)\right|^2 = \frac{2l+1}{4\pi}\,, \tag{186}$$

which can be generalized to the tensorial case (Blanchet/Damour [49])

$$\sum_{S}\sum_{m=-j}^{j} \left|\widehat{Y}_S^{sl,jm}(\mathbf{n})\right|^2 = \frac{2j+1}{4\pi}\,, \tag{187}$$

with $\mathbf{n}$ being the radial unit vector, whose direction is defined by θ and ϕ.

Thorne [548] provides details on the definition and application of vectorial and tensorial spherical harmonics, e.g., solutions of Laplace's equation or the wave equation. His article also contains explicit formulae for the STF version of tensor spherical harmonics. The latter can be used to obtain post-Newtonian expressions for the multipole moments (mass and mass current) of isolated gravitational sources, where „isolated" means that its matter occupies a finite domain in space. Remark: STF methods are just one of several alternatives to relativistically generalize the closed-form scalar results to the vector and tensor cases (Damour/Iyer [114]).

The time-dependent multipole moments result from a decomposition of the field in question into various spherical harmonics (vectorial, tensorial), requiring not only a spatial integration on the source but also a time integration. Already for the scalar case, we could replace the simple density ρ, i.e., the product $G\rho$, in Poisson equation (80) by a more general source $S(\mathbf{x},t)$. Generalizing the Laplace operator Δ by the D'Alembert operator $\Box$ we get

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta \quad \Rightarrow \quad \Box V = -4\pi S \tag{188}$$

with

$$V(\mathbf{X},T) = \int \frac{S(\mathbf{x},U)}{|\mathbf{X}-\mathbf{x}|}\,\mathrm{d}^3x\,, \tag{189}$$

where $U = T - |\mathbf{X}-\mathbf{x}|/c = T - R/c$, representing a time delay, indicates a finite velocity of propagation, e.g., finite speed of gravity. Therefore, V in this case is called a retarded potential. In the exterior of the source one can apply a multipole expansion, generalizing equation (177),

$$V(\mathbf{X},T) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\,\partial_L\left(\frac{F_L(U)}{R}\right). \tag{190}$$

The STF multipole moments $F_L(U)$ comprise time averages $\overline{S}_l(\mathbf{x},U)$, which are weighted by $\widehat{x}^L := x^{\langle i_1}\cdots x^{i_l\rangle}$, such that

$$F_L(U) = \int \widehat{x}^L \overline{S}_l(\mathbf{x},U)\,\mathrm{d}^3x \tag{191}$$

with $(r = |\mathbf{x}|)$

$$\overline{S}_l(\mathbf{x},U) = \int_{-1}^{1} \underbrace{\frac{(2l+1)!!}{2^{l+1}l!}\left(1-z^2\right)^l}_{=:\,\delta_l(z)} S\big(\mathbf{x},U+rz/c\big)\mathrm{d}z, \tag{192}$$

where a tilde may be used to denote z-retardation, e.g. $\widetilde{S} = S\big(\mathbf{x},U+rz/c\big)$.

Damour/Iyer [114] cite a series expansion, that can be applied in case of post-Newtonian expansions of gravitational fields:

$$\overline{S}_l(\mathbf{x},U) = \sum_{p=0}^{\infty} \frac{(2l+1)!!}{(2p)!!(2l+2p+1)!!}\left(\frac{r}{c}\right)^{2p} \frac{\partial^{2p}}{\partial U^{2p}}\,S(\mathbf{x},U). \tag{193}$$

The idea of STF multipole moments equally applies to Einstein's field equations (73), where the source is given by a tensor field T_{ij} and not by a scalar field anymore. Linearization via $g_{ij} = \eta_{ij} + h_{ij}$ enables the following approximative calculation of the Christoffel symbols (16) (Kopejkin et al. [299])

$$\Gamma^l_{ij} = \tfrac{1}{2}\big(\eta^{lk} - h^{lk}\big)\big(\partial_j h_{ik} + \partial_i h_{jk} - \partial_k h_{ij}\big) + \cdots \,, \tag{194}$$

and by making use of the harmonic gauge (147) leads to the following new field equations

$$\Box \bar{h}_{ij}(\mathbf{X},T) = -\frac{16\pi G}{c^4}\,T_{ij}(\mathbf{X},T), \tag{195}$$

with $\bar{h}_{ij} = h_{ij} - \frac{1}{2}h\,\eta_{ij}$. Accordingly, the multipole expansions now read

$$\begin{aligned}
\bar{h}^{00}(\mathbf{X},T) &= \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\left(\frac{F_L(U)}{R}\right),\\
\bar{h}^{0i}(\mathbf{X},T) &= \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\left(\frac{G_{iL}(U)}{R}\right),\\
\bar{h}^{ij}(\mathbf{X},T) &= \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\left(\frac{H_{ijL}(U)}{R}\right),
\end{aligned} \tag{196}$$

where

$$F_L(U) = \iint\limits_{-1}^{\;\;1}\widehat{x}^L\delta_l(z)\,\widetilde{T}^{00}\,\mathrm{d}z\,\mathrm{d}^3x\,,\quad G_{iL}(U) = \iint\limits_{-1}^{\;\;1}\widehat{x}^L\delta_l(z)\,\widetilde{T}^{0i}\,\mathrm{d}z\,\mathrm{d}^3x\,,\quad H_{ijL}(U) = \iint\limits_{-1}^{\;\;1}\widehat{x}^L\delta_l(z)\,\widetilde{T}^{ij}\,\mathrm{d}z\,\mathrm{d}^3x\,. \tag{197}$$

The quantities $G_{iL}(U)$ and $H_{ijL}(U)$, representing gravitoelectric and gravitomagnetic tidal moments, are reducible and can be decomposed into three and six irreducible pieces, respectively (Damour/Iyer [114]). Eventually, the authors introduce new STF tensors, notably (so-called Blanchet-Damour) mass multipole moments $M_L(U)$ ($l \geq 0$) and spin multipole moments $S_L(U)$ ($l \geq 1$) whose definition, especially in case of fast rotations, is related to the post-Minkowskian approximation scheme. The moments M_L in the Cartesian language are equivalent to the set of classical spherical harmonic coefficients c_{lm} and s_{lm}.

After a gauge transformation which preserves the harmonicity condition (147), the tensor $\bar{h}^{ij}$ can be brought to a canonical form $\bar{h}^{ij}_{\text{can}}$. In doing so its components can be expressed similar to equations (196), but now solely by means of $M_L(U)$, $S_L(U)$ and its time derivatives.

In general relativity, due to the non-linearity and tensorial character of the gravitational interaction, the definition of the multipole moments gets more complicated than in the Newtonian theory of gravitation. The existing gauge freedom makes the multipolar decomposition of the gravitational field coordinate-dependent, which ultimately affects any subsequent physical interpretation. There exist exact closed-form expressions for the multipole moments. Damour et al. [115], based on the theory of distributions in order to extract the non-divergent core of some other multipole moments given by Thorne [548], define (remember equations (160))

$$\begin{aligned}
M_L(T) &= \int\widehat{X}^L\Sigma\,\mathrm{d}^3X + \frac{1}{2(2l+3)\,c^2}\frac{\mathrm{d}^2}{\mathrm{d}T^2}\int\widehat{X}^L\Sigma\,\mathbf{X}^2\,\mathrm{d}^3X - \frac{4(2l+1)}{(l+1)(2l+3)\,c^2}\frac{\mathrm{d}}{\mathrm{d}T}\int\widehat{X}^{aL}\Sigma^a\,\mathrm{d}^3X \quad (l \geq 0)\,,\\
S_L(T) &= \int\varepsilon^{ab\langle c_l}\widehat{X}^{L-1\rangle a}\Sigma^b\,\mathrm{d}^3X \quad (l \geq 1)\,.
\end{aligned} \tag{198}$$

To check the validity of these equations we compare low-order moments with standard expressions from non-STF approaches. Exemplarily, in case of $l = 0$ the general gravitational multipole expression for M_L first reduces to

$$M = \frac{G}{c^2}\iint\limits_{-1}^{\;\;1}\delta_0\big(\widetilde{T}^{00} - zn_a\widetilde{T}^{0a}\big)\,\mathrm{d}z\,\mathrm{d}^3x\,, \tag{199}$$

and after using conservation equations for $\widetilde{T}^{00}$ and $\widetilde{T}^{0a}$ one gets with

$$M = \frac{G}{c^2}\int T^{00}(\mathbf{x},t)\,\mathrm{d}^3x \tag{200}$$

the standard expression for the total mass (mass monopole). For simplicity some authors put $G = 1$ and $c = 1$. With these settings, and similarly to M the mass dipole expression (case $l = 1$) reduces to

$$M_i = \iint_{-1}^{1} \delta_1 \Big(x^i \big(\widetilde{T}^{00} - z n_a \widetilde{T}^{0a}\big) - r z \big(\widetilde{T}^{0i} - z n_b \widetilde{T}^{ib}\big)\Big) \mathrm{d}z\, \mathrm{d}^3 x\,. \tag{201}$$

Again one incorporates conservation equations for the four-momentum. In addition, we can apply the conservation of relativistic angular momentum. Eventually one finds the mass dipole expression in standard form:

$$M_i = \int x^i T^{00}(\mathbf{x}, t)\, \mathrm{d}^3 x \tag{202}$$

If GCRS's origin coincides with Earth's post-Newtonian center of mass then the mass dipole of Earth vanishes. In this manner one could generate simplified expressions of higher-order mass multipole moments $M_{i_1 i_2 \ldots}$, too. The same holds true for spin multipole moments. For instance, the final expression for the total spin (case $l = 1$) reads (Damour/Iyer [114])

$$S_i = \int \varepsilon_{iab}\, x_a T^{0b}(\mathbf{x}, t)\, \mathrm{d}^3 x. \tag{203}$$

Klioner et al. [290] discuss the mathematical framework of (rigid) rotation in general relativity based upon multipole moments. If several bodies are involved or if one wants to indicate a specific body, moment symbols are to be completed by a corresponding (upper) index, i.e. M_L^A and S_L^A with $A = \oplus$ in case of Earth.

30.7 Earth's metric potentials in relativistic mass and spin multipole moments

The local parts of Earth's metric potentials, cf. equations (161), can now be rewritten as (Damour et al. [118])

$$\begin{aligned}
W_\oplus(T, \mathbf{X}) &= G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, \partial_L \left(\frac{M_L^\oplus(T \pm R/c)}{R} \right) + O\big(c^{-4}\big), \\
W_\oplus^a(T, \mathbf{X}) &= -G \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \left\{ \partial_{L-1} \left(\frac{\dot{M}_{aL-1}^\oplus(T \pm R/c)}{R} \right) + \frac{l}{l+1}\, \varepsilon_{abc}\, \partial_{bL-1} \left(\frac{S_{cL-1}^\oplus(T \pm R/c)}{R} \right) \right\} + O\big(c^{-2}\big)
\end{aligned} \tag{204}$$

with $R = \sqrt{\delta_{ij} X^i X^j} = |\mathbf{X}|$ and $f(T \pm R/c) = \big(f(T + R/c) + f(T - R/c)\big)/2$ as the average, which indicates the time-symmetric solution with spherical symmetry of the wave equation.

Retaining only T as the argument of the multipole moments one gets the equivalent formulas (Soffel et al. [512])

$$\begin{aligned}
W_\oplus(T, \mathbf{X}) &= G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left(M_L^\oplus(T)\, \partial_L \frac{1}{R} + \frac{1}{2c^2}\, \ddot{M}_L^\oplus(T)\, \partial_L R \right) + O\big(c^{-4}\big), \\
W_\oplus^a(T, \mathbf{X}) &= -G \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \left\{ \dot{M}_{aL-1}^\oplus(T)\, \partial_{L-1} \frac{1}{R} + \frac{l}{l+1}\, \varepsilon_{abc}\, S_{cL-1}^\oplus(T)\, \partial_{bL-1} \frac{1}{R} \right\} + O\big(c^{-2}\big).
\end{aligned} \tag{205}$$

30.8 Transformation between global and local reference systems

The tidal external potentials in equations (159) result from a superposition of several sources (linear split):

$$W_{\mathrm{tidal}}(T, \mathbf{X}) = \sum_{B \neq A} W_{BA}(T, \mathbf{X})\,, \quad W_{\mathrm{tidal}}^a(T, \mathbf{X}) = \sum_{B \neq A} W_{BA}^a(T, \mathbf{X})\,, \tag{206}$$

where the subindex BA denotes the influence of the source B on the primary body A. Again, we may set $A = \oplus$ in case of Earth as primary body.

One switches from local-frame multipole moments to global-frame multipole-like moments (Damour et al. [116]) by making use of a coordinate transformation between global and local reference systems, e.g. BCRS and GCRS. Formally, we could express such a transformation by (Klioner/Soffel [288])

$$\begin{aligned}
T &= t - \frac{1}{c^2}\big(A(t) + v_\oplus^i r_\oplus^i\big) + \frac{1}{c^4}\Big(B(t) + B^i(t) r_\oplus^i + B^{ij}(t) r_\oplus^i r_\oplus^j + C(t, \mathbf{x})\Big) + O\big(c^{-5}\big), \\
X^a &= R_j^a(t) \left(r_\oplus^j + \frac{1}{c^2} \Big(\tfrac{1}{2}\, v_\oplus^j v_\oplus^k r_\oplus^k + D^{jk}(t) r_\oplus^k + D^{jkl}(t) r_\oplus^k r_\oplus^l \Big) \right) + O\big(c^{-4}\big)
\end{aligned} \tag{207}$$

with $r^i_\oplus = x^i - x^i_\oplus(t)$, rotational matrix $R^a_j(t)$, and a number of additional functions, that all remain to be determined. By a simple choice one may globally fix the axes such that geocentric spatial coordinate lines are aligned with respect to the barycentric ones, i.e. $R^a_j = \delta_{aj}$. In doing so we get a geocentric kinematically non-rotating frame but have to take relativistic Coriolis forces into account. The effacement of the Coriolis effect in the geocentric frame would require the choice of a particular time dependence of the rotational matrix, which leads to a dynamically non-rotating frame (Damour et al. [118]).

Equation (207) can be used to transform the metric (148) into the metric (158), e.g., by matching of the metric tensors via the Jacobian $\partial X^\mu/\partial x^\varepsilon$, i.e.,

$$g_{\varepsilon\lambda}(t,\mathbf{x}) = \frac{\partial X^\mu}{\partial x^\varepsilon}\frac{\partial X^\nu}{\partial x^\lambda}\,\mathcal{G}_{\mu\nu}(T,\mathbf{X})\,. \tag{208}$$

The determination of the unknown functions $A, B, B^i, B^{ij}, C, D^{jk}, D^{jkl}$, and R^a_j can be achieved via the matching technique as outlined in Klioner/Soffel [288], cf. equations (225) and (226) in § 30.10.

Denoting the rotation matrix of the B-frame with respect to the A-frame by $R^{ab}_{BA} = R^{ia}_A R^{ib}_B$, the global-system velocity of the B-frame with respect to the A-frame by $v^i_{BA} = v^i_B - v^i_A$, and the projections of $\mathbf{v}_{BA}$ with respect to the local frames B and A by $V^b_{BA/B} = R^{ib}_B v^i_{BA}$ and $V^a_{BA/A} = R^{ia}_A v^i_{BA}$, one can write approximatively

$$\begin{aligned}
W_{BA}(T_A,\mathbf{X}_A) &= W_B(T_B,\mathbf{X}_B) + \frac{2}{c^2}\,\mathbf{v}^2_{BA}W_B(T_B,\mathbf{X}_B) + \frac{4}{c^2}V^b_{BA/B}W^b_B(T_B,\mathbf{X}_B),\\
W^a_{BA}(T_A,\mathbf{X}_A) &= V^a_{BA/A}W_B(T_B,\mathbf{X}_B) + R^{ab}_{BA}W^b_B(T_B,\mathbf{X}_B),
\end{aligned} \tag{209}$$

where $T_A = X^0_A/c$ and $T_B = X^0_B/c$ are the local systems' coordinate times corresponding to space-time points X^i_A and X^i_B, respectively. The various above mentioned velocities and rotation matrices bear these time arguments depending on indices A or B. All individual potential terms on the right hand side of equations (209) are expressible in terms of corresponding multipole moments, for instance M^B_L or M^{BA}_L (Damour et al. [116]).

In terms of global (barycentric) coordinates one gets (Damour et al. [118])

$$\begin{aligned}
W_{BA}(t,\mathbf{x}) &= \frac{\mu_B}{r_B}\left(1+\frac{2}{c^2}\big(\mathbf{v}_B-\mathbf{v}_A\big)^2-\frac{1}{c^2}\,w_{\mathrm{ext}B}(t,\mathbf{x}_B)-\frac{1}{2c^2}\big((\mathbf{n}_B\cdot\mathbf{v}_B)^2+\mathbf{a}_B\cdot\mathbf{r}_B\big)\right),\\
W^a_{BA}(t,\mathbf{x}) &= \frac{\mu_B}{r_B}\,R^{ia}_A v^i_{BA}
\end{aligned} \tag{210}$$

with $\mathbf{x}_B(t)$ denoting the barycentric position vector of the B-frame origin, e.g. the center of mass of body B. The remaining symbols are $\mathbf{r}_B = \mathbf{x}-\mathbf{x}_B$, $r_B = |\mathbf{r}_B|$, $\mathbf{n}_B = \mathbf{r}_B/r_B$, $\mathbf{v}_B = \mathrm{d}\mathbf{x}_B/\mathrm{d}t$, $\mathbf{a}_B = \mathrm{d}\mathbf{v}_B/\mathrm{d}t$, and $\mu_B = GM_B$.

30.9 External and tidal potentials in post-Newtonian approximation

Closed form expressions for the resulting tidal potentials W_{tidal} and W^a_{tidal} are given in Klioner/Voinov [286]. Alternatively, one can introduce relativistic tidal moments (gravitational gradients) G_L and H_L which can, similarly to the linearly superimposed external potential (inertial plus tidal parts), be decomposed into $N+1$ individual contributions, if N bodies are involved.

Defining the external gauge-invariant gravito-electric and gravito-magnetic fields (playing an important role for the equations of motion)

$$E_{\mathrm{ext}\,a}(T,\mathbf{X}) := \partial_a W_{\mathrm{ext}} + \frac{4}{c^2}\,\partial_T W^a_{\mathrm{ext}} \quad \text{and} \quad B_{\mathrm{ext}\,a}(T,\mathbf{X}) := -4\varepsilon_{abc}\,\partial_b W^c_{\mathrm{ext}}, \tag{211}$$

the corresponding post-Newtonian tidal moments are given by (Damour et al. [115])

$$G_L(T) := \partial_{\langle L-1}E_{\mathrm{ext}\,a_l\rangle}(T,\mathbf{X})\Big|_{X^a=0}, \qquad H_L(T) := \partial_{\langle L-1}B_{\mathrm{ext}\,a_l\rangle}(T,\mathbf{X})\Big|_{X^a=0} \qquad \text{for } (l\geq 1)\,. \tag{212}$$

For $l=0$ we have an additional datum (representing another degree of freedom) $G(T) := W_{\mathrm{ext}}(T,\mathbf{0}) + O(c^{-2})$, i.e., a monopole tidal moment, which can be gauged away by proper normalization. Remark: in all formulas we forgo the explicit writing of a gauge function (Soffel et al. [512])

$$\Lambda = G\sum_{l\geq 0}\frac{(-1)^l}{(l+1)!}\frac{2l+1}{2l+3}\int \widehat{X}^{aL}\Sigma^a\,\mathrm{d}^3X\,\partial L\frac{1}{R}\,, \tag{213}$$

because only its temporal and spatial partial differentials, i.e. $\Lambda_{,T}$ and $\Lambda_{,a}$, enter the final formulas for the potentials or the metric tensors. These gauge terms are of order $O(c^{-4})$ and are therefore negligible for most applications with the exception of highly accurate time scale transformations. For clock rate applications, as in chronometric leveling for instance, these terms are also expendable due to its small magnitude which is much less than 10^{-18}.

As a remark, which is especially valid for alternative theories to the general theory of relativity, due to the absence of clearly formulated field equations in the framework of the PPN formalism, the gauge function leaves too many degrees of freedom that can not be uniquely fixed. This leads to a researcher-dependent ambiguity in the interpretation of relativistic effects in GCRS (local reference) frames (Kopejkin et al. [299]).

Conversely to equation (212), the fields $E_{\mathrm{ext}\,a}$ and $B_{\mathrm{ext}\,a}$ as well as the external potentials itself can be expanded as series in powers of the local spatial coordinates, e.g.

$$W_{\mathrm{ext}} = \sum_{l\geq 0} \frac{1}{l!} G_L \widehat{X}^L + O(c^{-2}). \tag{214}$$

The tidal moments are directly related to the multipole moments. The tidal-dipole moment ($l=1$) can be written as (Damour et al. [118])

$$G_a = -\sum_{l\geq 2} \frac{1}{l!} \frac{M_L}{M} G_{aL} + O(c^{-2}). \tag{215}$$

For higher order tidal moments the explicit expressions become quite complex. Even under the simplifying assumption that all external bodies can be regarded as mass monopoles, the approximative post-Newtonian tidal-quadrupole matrix ($l=2$) reads

$$\begin{aligned} G_{ab} = \sum_{B\neq A} R_a^i R_b^j \frac{3\mu_B}{r_{AB}^3} \Big(n_{AB}^{\langle ij\rangle} + \frac{1}{c^2} \Big(n_{AB}^{\langle ij\rangle} \big(2\mathbf{v}_{AB}^2 - 2w_{\mathrm{ext}A}(t,\mathbf{x}_A) - w_{\mathrm{ext}B}(t,\mathbf{x}_B) - \tfrac{5}{2}(\mathbf{n}_{Ab}\cdot\mathbf{v}_B)^2 - \tfrac{1}{2}\,\mathbf{a}_B\cdot\mathbf{r}_{AB}\big) + \\ + a_{AB}^{\langle i} r_{AB}^{j\rangle} + v_{AB}^{\langle i} v_{AB}^{j\rangle} - 2(\mathbf{n}_{AB}\cdot\mathbf{v}_{AB})\, n_{AB}^{\langle i} v_{AB}^{j\rangle} - (\mathbf{n}_{AB}\cdot\mathbf{v}_A)\, n_{AB}^{\langle i}\big(v_A^{j\rangle} - 2v_B^{j\rangle}\big)\Big)\Big) + O(c^{-4}). \end{aligned} \tag{216}$$

For even higher tidal moments ($l>2$) the Newtonian limit may be sufficient, such that

$$G_L = R_{a_1}^{Ai_1} \cdots R_{a_l}^{Ai_l} \partial_L w_{\mathrm{ext}A}(t,\mathbf{x}_A) + O(c^{-2}) \tag{217}$$

with

$$w_{\mathrm{ext}A}(t,\mathbf{x}_A) = \sum_{B\neq A} \frac{\mu_B}{r_B} + O(c^{-2}). \tag{218}$$

Regarding the Newtonian case, already the Newtonian tidal potential (effective local potential) (167) may be written as a tidal expansion, explicitly

$$W_{\mathrm{tidal}}^{\mathrm{Newton}}(T,\mathbf{X}) = G_i^{\oplus} X^i + \frac{1}{2!} G_{ij}^{\oplus} X^i X^j + \cdots + \frac{1}{l!} G_{i_1\cdots i_l}^{\oplus} X^{i_1} \cdots X^{i_l} + \cdots \tag{219}$$

with the Newtonian tidal moments (Newtonian gravitational gradients)

$$G_i^{\oplus}(t) := \partial_i w_{\mathrm{ext}}(t,\mathbf{x}_{\oplus}) - \frac{\mathrm{d}^2 x_{\oplus}^i}{\mathrm{d}t^2} \quad (l=1) \qquad \text{and} \qquad G_L^{\oplus}(t) := \partial_L w_{\mathrm{ext}}(t,\mathbf{x}_{\oplus}) \quad (l\geq 2)\,. \tag{220}$$

The Earth's barycentric equations of motion under the influence of external bodies can thus be expressed by

$$M_{\oplus}\left(\frac{\mathrm{d}^2 x_{\oplus}^i}{\mathrm{d}t^2} - \partial_i w_{\mathrm{ext}}(t,\mathbf{x}_{\oplus})\right) = -M_{\oplus} G_i^{\oplus} = \sum_{l\geq 1} \frac{1}{l!} M_L^{\oplus} G_{iL}^{\oplus}\,, \tag{221}$$

where we used equation (215). Alternatively one gets (Damour et al. [115])

$$M_{\oplus} \frac{\mathrm{d}^2 x_{\oplus}^i}{\mathrm{d}t^2} = G \sum_{B\neq\oplus} \sum_{l\geq 0} \sum_{k\geq 0} \frac{(-1)^k}{l!k!} M_L^{\oplus} M_K^B\, \partial_{iLK}^{\oplus} \left(\frac{1}{|\mathbf{x}_{\oplus} - \mathbf{x}_B|}\right), \tag{222}$$

applying the condensed notation, i.e., $L := i_1 \cdots i_l$, $K := j_1 \cdots j_k$, and $\partial_i^{\oplus} := \partial/\partial x_{\oplus}^i$. Regarding the summation indices, because of the Newtonian center of mass definition, we get for the case $l=1$ zero mass-dipole moments ($M_i^{\oplus}(t) = 0$) and thus all terms with $l=1$ or $k=1$ vanish.

Coming back to the post-Newtonian case, if we need higher accuracy than implied by equation (214), also time derivatives of the tidal moments have to be taken into account, namely (Damour et al. [116])

$$\begin{aligned} W_{\text{ext}} &= \sum_{l\geq 0} \frac{1}{l!}\Big(G_L \widehat{X}^L + \frac{1}{2(2l+3)\,c^2}\,\mathbf{X}^2 \ddot{G}_L \widehat{X}^L\Big) + O\big(c^{-4}\big), \\ W^a_{\text{ext}} &= \sum_{l\geq 0} \frac{1}{l!}\Big(-\frac{2l+1}{(l+1)(2l+3)}\,\dot{G}_L \widehat{X}^{aL} + \frac{l}{4(l+1)}\,\varepsilon_{abc}\,H_{cL-1}\widehat{X}^{bL-1}\Big) + O\big(c^{-2}\big). \end{aligned} \tag{223}$$

The dominant term in the expansion (168) of W_{tidal} is the quadratic term ($l = 2$), which is by comparison with equation (223) equivalent to (Soffel et al. [512])

$$W_{\text{tidal}}\big|_{l=2} = \tfrac{1}{2} G_{ab} X^a X^b. \tag{224}$$

To emphasize, here G_{ab} is not a (local) metric tensor but the post-Newtonian tidal quadrupole matrix (216).

In the Newtonian approximation the Taylor series expansion for the gravitational potential of the tidal force usually starts from the quadratic term because the monopole and dipole external multipole moments are not physically associated with the tidal force. As we have seen before, in the framework of general relativity, the origin of tidal forces can be explained by means of the Riemannian curvature tensor, i.e., the second derivatives of the metric tensor. The monopole-dipole effacing property of the external gravitational field can always be extended from the Newtonian theory of gravity to the post-Newtonian approximation by a suitable coordinate transformation on the space-time manifold (Kopejkin et al. [299]). In other words, the gravitational field can be reduced to a pure tidal field based on Einstein's equivalence principle. Remark: this principle would be violated by a scalar-tensor theory of gravity and lead to the Nordtvedt effect. Regarding the gauge degree of freedom, a physically meaningful choice implies that any gauge-dependent external multipoles which do not carry out gravitational degrees of freedom are eliminated by infinitesimal coordinate transformations. The functional form of the Riemannian curvature tensor is invariant with respect to such a transformation. The residual gauge freedom can be used differently to take advantage of a certain property of the metric tensor. For instance, one may stipulate the pure spatial part of the local metric tensor to be proportional to the unit matrix.

30.10 Transformation between BCRS and GCRS

Equation (210) indicates that we can transform between local and barycentric gravitational potentials based on a given spatio-temporal coordinate transformation. In practice, choosing GCRS and BCRS as reference systems, one applies the time scales $T = TCG$ and $t = TCB$, respectively. Denoting GCRS coordinates by X^α and BCRS coordinates by x^μ, its transformations are equivalently given either by $X^\alpha = X^\alpha(t, x^i)$ (BK-approach) or by $x^\mu = x^\mu(T, X^a)$ (DSX-approach).

The IAU resolutions recommend, similarly to equations (207), the following explicit formulas (Soffel et al. [512])

$$\begin{aligned} T &= t - \frac{1}{c^2}\big(A(t) + v^i_\oplus r^i_\oplus\big) + \frac{1}{c^4}\Big(B(t) + B^i(t) r^i_\oplus + B^{ij}(t) r^i_\oplus r^j_\oplus + C(t,\mathbf{x})\Big) + O\big(c^{-5}\big), \\ X^a &= \delta_{ai}\bigg(r^i_\oplus + \frac{1}{c^2}\Big(\tfrac{1}{2}\, v^i_\oplus v^j_\oplus r^j_\oplus + w_{\text{ext}}(t,\mathbf{x}_\oplus)\, r^i_\oplus + r^i_\oplus a^j_\oplus r^j_\oplus - \tfrac{1}{2}\, a^i_\oplus r^2_\oplus\Big)\bigg) + O\big(c^{-4}\big) \end{aligned} \tag{225}$$

with

$$\begin{aligned} \dot{A}(t) &= \tfrac{1}{2}\, v^2_\oplus + w_{\text{ext}}(t,\mathbf{x}_\oplus)\,, \\ \dot{B}(t) &= -\tfrac{1}{8}\, v^4_\oplus - \tfrac{3}{2}\, v^2_\oplus w_{\text{ext}}(t,\mathbf{x}_\oplus) + 4 v^i_\oplus w^i_{\text{ext}}(t,\mathbf{x}_\oplus) + \tfrac{1}{2}\, w^2_{\text{ext}}(t,\mathbf{x}_\oplus)\,, \\ B^i(t) &= -\tfrac{1}{2}\, v^2_\oplus v^i_\oplus + 4 w^i_{\text{ext}}(t,\mathbf{x}_\oplus) - 3 v^i_\oplus w_{\text{ext}}(t,\mathbf{x}_\oplus)\,, \\ B^{ij}(t) &= -v^i_\oplus \delta_{aj}\, Q^a + 2\frac{\partial}{\partial x^j}\, w^i_{\text{ext}}(t,\mathbf{x}_\oplus) - v^i_\oplus \frac{\partial}{\partial x^j}\, w_{\text{ext}}(t,\mathbf{x}_\oplus) + \tfrac{1}{2}\,\delta^{ij}\,\dot{w}_{\text{ext}}(t,\mathbf{x}_\oplus)\,, \\ C(t,\mathbf{x}) &= -\tfrac{1}{10}\, r^2_\oplus\big(\dot{a}^i_\oplus r^i_\oplus\big). \end{aligned} \tag{226}$$

The scalar and vectorial external potentials are to be computed again by linear superposition, according to equation (153) but without Earth's contribution, i.e. for the summation index we have $A \neq \oplus$. For individual contributions of third bodies one has to take the integrals following (152), as mentioned before. The quantity Q^a was already given by equation (162). Finally, Earth's local scalar and vectorial gravitational potentials are related to the barycentric ones via

$$\begin{aligned} W_\oplus(T,\mathbf{X}) &= w_\oplus(t,\mathbf{x})\Big(1 + \frac{2}{c^2}\, v^2_\oplus\Big) - \frac{4}{c^2}\, v^i_\oplus w^i_\oplus(t,\mathbf{x}) + O\big(c^{-4}\big), \\ W^a_\oplus(T,\mathbf{X}) &= \delta_{ai}\big(w^i_\oplus(t,\mathbf{x}) - v^i_\oplus w_\oplus(t,\mathbf{x})\big) + O\big(c^{-2}\big). \end{aligned} \tag{227}$$

In general, a consistent introduction of a local coordinate system requires the knowledge of the nature of every involved fundamental field within the metric tensor because each one may show a specific behavior under coordinate transformations. The solution for the field equations is to be found directly in this local coordinate system and must be matched to the solution of the same field equations in the global coordinate system. The correct relativistic space-time transformation between both systems then results from a combined use of the transformation laws for the fields and the metric tensors, respectively (Kopejkin et al. [299]).

Remark: transformation (225) represents an approximative Lorentz transformation of special relativity extended by gravitational fields and acceleration terms (Brumberg/Kopejkin [73], Soffel et al. [512]).

As mentioned before, nominal values of multipole moments (198) as being parameters of the gravitational field are fitted to observations. This idea corresponds to the conventional (space) geodetic gravity field determination approach. Earth's scalar gravitational potential $W_\oplus(T, \mathbf{X})$ may equivalently be written in the classical form as

$$W_\oplus = \frac{\mu_\oplus}{R}\left(1 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a_\oplus}{R}\right)^l P_{lm}(\cos\theta)\big(\mathcal{C}^\oplus_{lm}\cos m\phi + \mathcal{S}^\oplus_{lm}\sin m\phi\big)\right) + O\big(c^{-4}\big), \tag{228}$$

where θ and ϕ are the polar angles corresponding to the spatial GCRS-related coordinates X^a. As usual, omitting any terms of degree one imposes the constraint, that the coordinate system is centered at the body's center of mass. The spherical harmonic coefficients, now depending on T and $R = |\mathbf{X}|$, are defined by (cf. equations (43) and (44) in Soffel et al. [512])

$$\mathcal{C}^\oplus_{lm}(T,R) = C^\oplus_{lm}(T) - \frac{1}{2(2l-1)\,c^2}\,R^2\ddot{C}^\oplus_{lm}(T)\,, \qquad \mathcal{S}^\oplus_{lm}(T,R) = S^\oplus_{lm}(T) - \frac{1}{2(2l-1)\,c^2}\,R^2\ddot{S}^\oplus_{lm}(T)\,. \tag{229}$$

Again, the set of the coefficients $C^\oplus_{lm}(T)$ and $S^\oplus_{lm}(T)$ is, to sufficient accuracy, equivalent to the set of the mass multipole moments $M^\oplus_L(T)$. They are related to the approximatively constant classical spherical harmonic coefficients c_{lm} and s_{lm}, which refer to a terrestrial system that rotates with the Earth (ECEF, e.g., ITRF), by time-dependent transformations. Soffel et al. [512] discuss the orders of magnitude of individual terms in the series expansion. As a result they state that, due to its tiny effects of order 10^{-27}, the second time derivatives from equation (229) can safely be ignored at present.

30.11 Remarks on various spin-related terms

Likewise, Soffel et al. [512] provide a quantitative appraisal of the vectorial gravitational potential. Most of the practical applications certainly will not require an extensive series expansion of $W^a_\oplus(T, \mathbf{X})$. Instead, it may be sufficient to use the approximative expression

$$W^a_\oplus(T,\mathbf{X}) = \frac{G}{2R^3}\big(\mathbf{S}_\oplus \times \mathbf{X}\big)^a, \tag{230}$$

where $\mathbf{S}_\oplus$ denotes Earth's spin vector, remember our preliminary discussion of the gravitomagnetic field in equations (112) till (120). As Soffel et al. [512] point out, it is of conceptual advantage to characterize $W^a_\oplus$ rather by the spin vector than by the angular velocity vector of the Earth, because it allows for the effective use of well-defined multipole moments which are independent of any theoretical assumptions about the (irregular) rotational motion of the Earth. Nonetheless, one can find earlier treatises that prefer the explicit use of Earth's angular momentum (e.g. Kusche [304]). In principle, Earth's gravitomagnetic field may be detected by orbiting superconducting gravity gradiometers (e.g. Gill et al. [209], Paik [400], Kopejkin et al. [299]).

For precise time scale comparisons it is necessary to account for the influence of major solar-system bodies like Jupiter and Saturn, as we know already from equations (142) and (143). Following the IAU 2000 resolutions on relativity, the transformation between TCB and TCG requires some additional spin-related terms. Considering the spin of celestial objects one naturally prefers the use of a global metric, i.e. the application of barycentric gravitational potentials. Based on equations (148) and (154) we may use the following alternative formulas, which are approximative (by decomposing and partial neglecting) and augmented (by additional spin-dipole moments) at the same time, for the metric tensor

$$\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}\Big(w_0 + w_L\Big) - \frac{2}{c^4}\Big(w_0^2 + \sum_A \Delta_A\Big) + O\big(c^{-5}\big),\\ g_{0i} &= -\frac{4}{c^3}\,w^i + O\big(c^{-5}\big),\\ g_{ij} &= \delta_{ij}\Big(1 + \frac{2}{c^2}\,w_0\Big) + O\big(c^{-4}\big), \end{aligned} \tag{231}$$

and, besides the expansion in higher order multipole moments $w_L(t, \mathbf{x})$, for the remaining potentials

$$
\begin{aligned}
w_0(t, \mathbf{x}) &= \sum_A \frac{\mu_A}{r_A}\,, \\
\Delta_A(t, \mathbf{x}) &= \frac{\mu_A}{r_A}\left(2v_A^2 - \sum_{B \neq A} \frac{\mu_B}{r_{BA}} - \frac{\left(r_A^k v_A^k\right)^2}{2r_A^2} - \frac{r_A^k a_A^k}{2}\right) + \frac{2G v_A^k \left(\mathbf{r}_A \times \mathbf{S}_A\right)^k}{r_A^3}\,, \qquad (232) \\
w^i(t, \mathbf{x}) &= \sum_A \left(\frac{\mu_A}{r_A}\, v_A^i - \frac{G\left(\mathbf{r}_A \times \mathbf{S}_A\right)^i}{2r_A^3}\right).
\end{aligned}
$$

Approximative spin values for Jupiter and Saturn are provided by the IAU resolutions, see Soffel et al. [512]. The mass monopoles ($l = 0$) of the individual bodies are included within the gravitational parameters μ.

30.12 Remarks on various kinds of mass-multipole moments

A generalization of the IAU 2000 resolutions on relativity may be achieved by the derivation of a scalar-tensor theory of gravity, i.e., the additional inclusion of phenomenological parameters by introduction of a scalar field besides the general relativistic tensorial field g_{α_β}.

Providing a detailed discussion on necessary steps for such a generalization is way beyond the scope of the present work. Here we only adumbrate that it requires the introduction of several sets of multipole moments, namely active mass multipoles I_L, scalar mass multipoles $\bar{I}_L$, and spin multipoles (Kopejkin et al. [299]). In order to satisfy various post-Newtonian laws of conservation, one has to linearly combine the different mass multipoles which eventually leads to the definition of a new set of conformal mass multipole moments $\tilde{I}_L$. Assuming an isolated astronomical N-body system, in the global frame there exists a simple algebraic relationship, e.g. in case of general relativity,

$$
I_L = \tilde{I}_L + \tfrac{1}{2}\, \bar{I}_L\,. \qquad (233)
$$

Similar relations apply for the case of individual sub-systems of bodies in the local frame. In practice, the conservation laws contain some simplifying assumptions. For instance, one neglects any loss of energy or momentum of the isolated system due to the withdrawing action of gravitational waves.

As with the several sets of (relativistic) mass multipole moments there exist several definitions of mass. The baryon (rest) mass is defined as the integral of the before mentioned baryon mass density ρ^* over the body's volume in local coordinates (in general a hypersurface of constant local time). Considering the relative velocity of the body, its internal elastic energy, as well as its Newtonian gravitational potential, one can derive the general relativistic post-Newtonian mass of the body (Will [586]). Regarding the conformal multipoles, this relativistic mass is altered by some additional terms towards a conformal mass. Besides that, any tests on the strong equivalence principle require the concept of an active mass of the body (Kopejkin et al. [299]).

The applicability and relative merits of different relativistic multipole definitions has yet to prove in practice. For the moment it seems advisable to implement the IERS recommendations and conventions as accurate as possible.

30.13 Gravitational potential knowledge and time transformation

Applications based on precise time and frequency measurements via modern atomic clocks or frequency standards require even more precise time coordinates and time transformation models. The capabilities of available or foreseeable instrumentation requires an accuracy level of at least 10^{-18} for the uncertainty in rate and fractions of a picosecond in amplitude for quasi-periodic terms.

The barycentric scalar gravitational potential can be decomposed into (Soffel et al. [512])

$$
w = w_0 + w_L - \frac{1}{c^2} \sum_A \Delta_A\,, \qquad (234)
$$

assuming that for earthbound clock experiments major solar system bodies like the Sun, Jupiter and Saturn have to be taken into account. On one hand, we treat all third bodies simply as mass monopoles such that w_L comprises (higher order) mass multipole moments ($l \geq 1$) only due to Earth. Therefore we may compute this quantity by using the first equation of (205) or equation (228) and by applying the transformation (227).

On the other hand, we may include the spinning of the massive solar-system objects to account for the Lense-Thirring effect. Such higher order effects are only of importance for extreme ambitious applications like relativity testing. Theoretically, the gravitational interaction of multipole moments of the primary body (e.g. Earth) with an external tidal field induced by other bodies leads to deviations of the primary body's (center-of-mass) world line from a geodesic. This had to be taken into account in precise numerical calculations (Kopejkin et al. [299]).

In practice however, due to the uncertainties in the mass multipole moments, which at present still lead to error levels that exceed the magnitude of all relativistic spin-related terms in Earth's vicinity, it is not necessary to account for these higher order effects at the moment. Furthermore, for earthbound clock readings it is sufficient to retain only the $\Delta_\oplus$-related term instead of all Δ_A's in the metric tensor (231).

In order to accurately account for all tidal effects, one finally will also have to incorporate any elastic deformations of the Earth's body, which give rise to Newtonian and post-Newtonian responses, too. Consequently, a relativistic theory of elasticity of a deformable body had to be developed (e.g. Xu et al. [607], [608], [609]). For some preliminary studies we will neglect those higher order effects.

Consequently, the transformation between proper time τ of the (optical atomic) clock and coordinate time (equating t with TCB) for now reads (Soffel et al. [512])

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 - \frac{1}{c^2}\Big(w_0 + w_L + \tfrac{1}{2}\,v^2\Big) + \frac{1}{c^4}\Big(\tfrac{1}{2}\,w_0^2 - \tfrac{3}{2}\,w_0 v^2 - \tfrac{1}{8}\,v^4 + 4v^i w^i + \Delta_\oplus\Big), \tag{235}$$

which is more accurate than equation (133).

In the same manner one can set up a refined equation for the difference $TCB - TCG$, which makes use of the external part of w_0 only, i.e., without the contributions of Earth itself. The evaluation comprises planetary ephemeris data which are expressed various time scales, depending on the institution or research group. Therefore, e.g., in case of using ephemeris time T_{eph}, a slightly modification of the resulting final formulas may become necessary (Soffel et al. [512]). Remark: today's major planetary ephemeris providers adopted different time scales (Folkner et al. [193], Pitjeva [423], Fienga et al. [183], [184]).

Comparing proper time against TCG is related to the GCRS metric tensor (158) and requires the accurate knowledge of its associated potentials W and W^a at the location of the clock. Earthbound applications are based on the terrestrial time scale TT and its several realizations. The transformation between TT and TCG depends on the nominal geoidal geopotential value W_0 (Burša et al. [78], [79]), not to be confused here with the gravitational potential W.

The uncertainty level in the determination process, that led to the current fixed value of W_0, will become insufficient for future highly precise time and frequency applications. Furthermore, it is not a trivial task to realize the (time variable) surface of the geoid with an accuracy, say to the *cm*-level, which is necessary for the exploitation of upcoming optical atomic clocks. Conversely, we may solve the inverse problem, i.e., determine potential differences and consequently heights directly by chronometric leveling.

30.14 Topocentric reference system connected to (earthbound) observation sites

The application of general relativity in geodesy comprises several reference frames. Incorporating space geodetic techniques implies the use of a (quasi-)inertial, i.e. barycentric, reference frame, whereas in Earth modeling we prefer a (dynamically) non-rotating geocentric reference frame.

Furthermore, any geodetic measurements at distributed observation stations are related to topocentric reference frames and its realizations (TRS). Of course, an earthbound station could also be replaced by a space-based station, e.g. a satellite. In this case one had to rename TRS by SRS (satellite reference system).

Having discussed the transformation between BCRS and GCRS already, it remains to treat the transformation between GCRS and TRS with due diligence, too. This is also required for the application of post-Newtonian gravimetry and gradiometry accompanied by (optical) atomic clock readings. Remark: simplifications arise if we assume clocks in rest with respect to Earth's surface.

Next, we summarize fundamental relations that are readily available (e.g. Kopejkin [295], Kopejkin et al. [299]). Let us first recap and substantiate the approximative local metric tensor (158):

$$\begin{aligned}
\mathcal{G}_{00} &= -1 + \frac{2}{c^2}\Big(W_\oplus(T,\mathbf{X}) + Q_i X^i + \tfrac{1}{2}\,Q_{ij}X^iX^j + \tfrac{1}{6}\,Q_{ijk}X^iX^jX^k\Big) + \frac{2}{c^4}\Big(\Phi_\oplus(T,\mathbf{X}) - W_\oplus^2(T,\mathbf{X}) - \tfrac{1}{2}\ddot{\Psi}_\oplus(T,\mathbf{X})\Big),\\
\mathcal{G}_{0a} &= -\frac{4}{c^3}\,W_\oplus^a(T,\mathbf{X})\,,\\
\mathcal{G}_{ab} &= \delta_{ab}\bigg(1 + \frac{2}{c^2}\Big(W_\oplus(T,\mathbf{X}) + Q_i X^i + \tfrac{1}{2}\,Q_{ij}X^iX^j + \tfrac{1}{6}\,Q_{ijk}X^iX^jX^k\Big)\bigg),
\end{aligned} \tag{236}$$

where, in accordance with equations (157), the essential potential functions are formally given by

$$
\begin{aligned}
W_\oplus &= G\int\limits_\oplus \frac{\rho^*(T,\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|}\,\mathrm{d}^3X', \\
W^a_\oplus &= G\int\limits_\oplus \frac{\rho^*(T,\mathbf{X}')\,V^a(T,\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|}\,\mathrm{d}^3X', \\
\Phi_\oplus &= G\int\limits_\oplus \frac{\rho^*(T,\mathbf{X}')\Big(\tfrac{3}{2}\,V^2(T,\mathbf{X}')+\Pi(T,\mathbf{X}')-W_\oplus(T,\mathbf{X}')\Big)+t^{kk}(T,\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|}\,\mathrm{d}^3X', \\
\Psi_\oplus &= -G\int\limits_\oplus \rho^*(T,\mathbf{X}')\,|\mathbf{X}-\mathbf{X}'|\,\mathrm{d}^3X'
\end{aligned}
\tag{237}
$$

with $\mathbf{V}=\mathrm{d}\mathbf{X}/\mathrm{d}T$ denoting the GCRS related velocity of Earth's matter, and t^{kk} representing the trace of the stress-energy tensor, which only in case of a perfect fluid simply reduces to $3p$, as assumed in equations (157).

The coupling between Earth's oblateness and the tidal gravitational octupole incorporates the conformal mass concept along with the spin multipole moments (Kopejkin et al. [299]), but it can be approximated in reference to Earth's baryon (rest) mass and second moment of inertia, namely

$$
M_\oplus=\int\limits_\oplus \rho^*(T,\mathbf{X})\,\mathrm{d}^3X \qquad\text{and}\qquad I^{ij}_\oplus=\int\limits_\oplus \rho^*(T,\mathbf{X})\,X^iX^j\,\mathrm{d}^3X, \tag{238}
$$

such that the inertial force, representing the indirect flattening effect from classical geodynamics, may be rewritten as

$$
M_\oplus Q_i=-\tfrac{1}{2}\,Q_{ijk}I^{jk}_\oplus. \tag{239}
$$

From equations (168) and (169) we know already that any higher order tidal gravitational multipole moment is obtained by partial differentiation of the external gravitational potential, i.e. $Q_L=w_{\mathrm{ext},L}$. For highly precise applications relativistic terms must not be neglected.

Outside the Earth the metric potentials (237) can be expressed in terms of internal multipole harmonics with respect to powers of $a_\oplus/R$ with $R=|\mathbf{X}|$, and $a_\oplus$ denoting Earth's mean radius. Neglecting higher order terms one gets, in accordance with equations (177), (178), (205) and (230), the specific expressions

$$
\begin{aligned}
W_\oplus &= \frac{\mu_\oplus}{R}+G\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\,M^\oplus_L\,\partial_L\Big(\frac{1}{R}\Big), &\qquad \Phi_\oplus &= \frac{G}{2R}\left(\int\limits_\oplus \rho^*\Big(V^2+2\Pi-W_\oplus\Big)\mathrm{d}^3X+\ddot{I}^{kk}_\oplus\right), \\
W^a_\oplus &= \frac{G}{2R^3}\Big((\mathbf{S}_\oplus\times\mathbf{X})^a+\dot{I}^{aj}_\oplus X^j\Big), & \Psi_\oplus &= -\mu_\oplus R-\frac{G}{3R}\,I^{kk}_\oplus-\frac{3}{2R^3}\,M^\oplus_{ij}X^iX^j.
\end{aligned}
\tag{240}
$$

with the following approximative formulas for the defining expressions (198)

$$
M^\oplus_L=\int\limits_\oplus \rho^*\,\widehat{X}^L\,\mathrm{d}^3X, \qquad S^\oplus_i=\int\limits_\oplus \rho^*\big(\mathbf{X}\times\mathbf{V}\big)^i\,\mathrm{d}^3X, \tag{241}
$$

where we only retained the spin dipole moment, i.e., the intrinsic angular momentum of the Earth $S^\oplus_i$ but an infinite number of Newtonian mass multipole moments. The latter are equivalent to the classical geopotential coefficients c_{lm} and s_{lm}. In case of axisymmetric mass distribution (only zonal coefficients $J_l=-c_{l0}$) one finds the simple relation (Kopejkin et al. [299])

$$
M^\oplus_L=M_\oplus r^l J_l\,\widehat{\mathbf{s}}^L, \tag{242}
$$

where $r=|\mathbf{x}-\mathbf{x}_\oplus|$, and $\mathbf{s}$ denotes the unit vector directed along the axis of rotation, i.e. $\mathbf{S}_\oplus=S_\oplus\,\mathbf{s}$ with $S^2_\oplus=\delta^{ij}S^\oplus_iS^\oplus_j$.

Being the proper reference frame of the observer, we replace T by τ in case of a (dynamically) non-rotating TRS. In accordance with the theory of general relativity one may use any coordinates to describe local measurements. Following the IAU recommendations we applied the harmonic gauge before. Consequently, as with the BCRS and GCRS, again we choose so-called harmonic coordinates $\xi^\alpha=(c\tau,\xi^i)$, which satisfy the homogeneous D'Alembert equation, i.e. $\Box\xi^\alpha=0$. This choice is convenient for our purposes because it leads to simplified practical calculations in case of our intended slow-motion and weak-field applications. The spatial coordinates ξ^i

can be interpreted as Euclidean coordinates from Newtonian theory. They are no longer harmonic if we introduce (rigid) rotations.

The observer (atomic clock) is assumed not to be affecting the gravitational field by itself, and its mass to be negligibly small. Thus, we can set any of its corresponding mass-describing functions (mass density, energy-density, stresses etc.) equal to zero. Denoting the TRS related metric tensor by $G_{\mu\nu}$, not to be confused here with the Einstein tensor (73)) or the tidal quadrupole matrix (216), we get

$$G_{\mu\nu}(\tau,\boldsymbol{\xi}) = \mathrm{diag}\Big(-1+\Xi(\tau,\boldsymbol{\xi}), 1+\Xi(\tau,\boldsymbol{\xi}), 1+\Xi(\tau,\boldsymbol{\xi}), 1+\Xi(\tau,\boldsymbol{\xi})\Big) \tag{243}$$

with (Kopejkin [295], Kopejkin et al. [299])

$$\Xi(\tau,\boldsymbol{\xi}) = \frac{2}{c^2}\Big(E_i(\tau)\,\xi^i + \tfrac{1}{2}\,E_{ij}(\tau)\,\xi^i\xi^j + \tfrac{1}{6}\,E_{ijk}(\tau)\,\xi^i\xi^j\xi^k\Big), \tag{244}$$

where the STF tensors E_L are taken at the TRS's origin as follows from the the solution of the homogeneous Laplace equations. In case of an earthbound system we have a non-vanishing inertial acceleration E_i of Earth's gravity with post-Newtonian corrections taken into account, and therefore TRS itself is a non-inertial system. Ground-based accelerometer measurements (gravimetry) of the force F^i acting upon a test mass m at TRS's origin will deliver information about the acceleration because $E^i = F^i/m$. For a satellite with drag-free control system (free falling experiment) we have $E_i = 0$ for the SRS, of course. As always, one can split this expression into a Newtonian and post-Newtonian part, i.e. $E_i = g_i^N + g_i^{PN}/c^2 + O(c^{-4})$, where the letter g implies a strong relation with the classical acceleration of gravity.

The tidal forces, related to E_{ij} and E_{ijk}, can be measured by means of gradiometry, again centering the instrument at the origin of the topocentric system (TRS or SRS). Post-Newtonian gravimetry and gradiometry are based on its respective classical counterparts but with consideration of non-negligible relativistic effects due to the improved sensitivity of contemporary instruments. Future devices will incorporate quantum engineering techniques but classical test masses are still common today, e.g. in superconducting gravity gradiometers. Measurements of the strength and direction of a gravitational field (gravimetry) and its differential acceleration over a unit distance (gradiometry) will prospectively be accompanied by (optical) atomic clock readings, thus accumulating the arsenal of instrumentation for relativistic geodesy. Successful application requires the availability of the final relationship between GCRS and TRS/SRS. In general, we will denote the origin of the TRS/SRS by a subindex T, and again any (atomic) clock-related quantity will be indicated by a subindex A.

30.15 Specific relations between geocentric time and proper time

The transformation between GCRS and TRS reads

$$\tau = T - \frac{1}{c^2}\Big(V(T) + V_T^k R_T^k\Big) - \frac{1}{c^4}\,S(T), \qquad \xi^i = R_T^i + \frac{1}{c^2}\Big(\tfrac{1}{2}\,V_T^i V_T^k + R^{ik}(T) + Z^{ik}(T)\Big)R_T^k \tag{245}$$

with $\mathbf{R}_T = \mathbf{X} - \mathbf{X}_T$, and $\mathbf{V}_T = \mathrm{d}\mathbf{X}_T/\mathrm{d}T$. Please note that $\mathbf{V}_T \neq \mathrm{d}\mathbf{R}_T/\mathrm{d}T$. Considering as a special case a clock A as being the observer (origin of the TRS), we get $\mathbf{R}_T = \mathbf{R}_A = \mathbf{0}$, and therefore equation (245) will simply provide the relationship between proper time τ and geocentric time $T = TCG$, namely

$$T - \tau = \frac{1}{c^2}\,V(T) + \frac{1}{c^4}\,S(T)\,. \tag{246}$$

In general, we have

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}T} &= \tfrac{1}{2}\,V_T^2 + W_\oplus + Q_i X_T^i + \tfrac{1}{2}\,Q_{ij}X_T^i X_T^j + \tfrac{1}{6}\,Q_{ijk}X_T^i X_T^j X_T^k\,,\\
\frac{\mathrm{d}S}{\mathrm{d}T} &= \tfrac{1}{8}\,V_T^4 + \tfrac{3}{2}\,V_T^2 W_\oplus - \tfrac{1}{2}\,W_\oplus^2 - 4V_T^a W_\oplus^a + \Phi_\oplus - \tfrac{1}{2}\,\ddot{\Psi}_\oplus\,,\\
\frac{\mathrm{d}R^{ab}}{\mathrm{d}T} &= 3V_T^{[a}W_\oplus^{,b]} - 4W_\oplus^{[a,b]} + V_T^{[a}E^{b]},\\
Z^{ab} &= \delta^{ab}\Big(W_\oplus + Q_i X_T^i + \tfrac{1}{2}\,Q_{ij}X_T^i X_T^j + \tfrac{1}{6}\,Q_{ijk}X_T^i X_T^j X_T^k\Big)
\end{aligned} \tag{247}$$

with all potential functions on the right hand side evaluated at $\mathbf{X}_T$. The matrices $R^{ab}(T)$ (referring to the spatial axes' precession/rotation) and $Z^{ab}(T)$ (related to the spatial coordinates' grid contraction) are anti-symmetric and symmetric, i.e., $R^{ab} = -R^{ba}$ and $Z^{ab} = Z^{ba}$, respectively. Again, one can derive formulas for the combined effect of the various precessional motions.

Putting the clock (or any other geodetic instrument) at rest with respect to the dynamically non-rotating GCRS, i.e., setting $\mathbf{V}_T = \mathbf{V}_A = \mathbf{0}$, resembles as a special case the form of expressions (133) or (136), (225), and (235),

$$T - \tau = \frac{1}{c^2} \int W_\oplus \, \mathrm{d}T + \frac{1}{c^4} \int \left(-\tfrac{1}{2} W_\oplus^2 + \Phi_\oplus - \tfrac{1}{2} \ddot{\Psi}_\oplus \right) \mathrm{d}T = \frac{1}{c^2} \int V_\oplus \, \mathrm{d}T, \tag{248}$$

where we introduced the generalized gravitational potential (Kopejkin et al. [299])

$$V_\oplus = W_\oplus + \frac{1}{c^2} \left(-\tfrac{1}{2} W_\oplus^2 + \Phi_\oplus - \tfrac{1}{2} \ddot{\Psi}_\oplus \right). \tag{249}$$

30.16 Post-Newtonian gravimetry and gradiometry

Gravimeter measurements alone only provide non-separable information on the Newtonian and post-Newtonian acceleration of gravity, because both

$$\begin{aligned} g_i^N &= V_{\oplus,i} - A_T^i + Q_i + Q_{ij} X_T^j + \tfrac{1}{2} Q_{ijk} X_T^j X_T^k \,, \\ g_i^{PN} &= 4\big(\dot{W}_\oplus^i + V_T^k W_{\oplus,k}^i - V_T^k W_{\oplus,i}^k\big) - 3\big(V_T^i \dot{W}_\oplus - A_T^i W_\oplus\big) - \\ &\quad - \tfrac{7}{2} V_T^i V_T^k W_{\oplus,k} + 2 V_T^2 W_{\oplus,i} - \tfrac{1}{2} V_T^i \big(A_T^k V_T^k\big) - V_T^2 A_T^i + g_j^N R^{ij} \end{aligned} \tag{250}$$

are composed of various superimposed effects. As before, $\mathbf{X}_T$ and $\mathbf{V}_T$ are the GCRS-related coordinates and velocity of the observer (gravimeter test mass in this case), and accordingly $\mathbf{A}_T = \mathrm{d}\mathbf{V}_T/\mathrm{d}T$ denotes its acceleration. Kopejkin et al. [299] discuss additional aspects when introducing specialized terrestrial reference frames, i.e. details on the consistent treatment of rotational effects, precession, nutation, etc. The second equation of (250) indicates a dependency on both potentials (Newtonian and gravitomagnetic), the observer's velocity and acceleration, as well as various correlations between all those quantities.

Besides gravimetry one can gain information on Earth gravity field by means of (relativistic) gradiometry. The basic principle is to precisely measure the deviation of world lines of various test masses. By means of a gradiometer at the origin of the TRS (local observer) one measures the quadrupole tidal field K_{ij} and its spatial derivative (octupole tidal field K_{ijk}), which directly relate to the Riemannian curvature tensor. Again separating Newtonian and post-Newtonian parts, one gets (neglecting higher order terms)

$$K_{ij} = -R_{0i0j} = K_{ij}^N + \frac{1}{c^2} K_{ij}^{PN}, \qquad K_{ijk} = -R_{0i0j,k} = K_{ijk}^N \tag{251}$$

with the individual tidal matrices (Kopejkin et al. [299])

$$K_{ij}^N = W_{\oplus,\langle ij \rangle} + 3Q_{ij} + 15 Q_{ijk} X_T^k, \qquad K_{ijk}^N = W_{\oplus,\langle ijk \rangle} + 15 Q_{ijk} \,, \qquad K_{ij}^{PN} = 2 K_{k\langle j}^N R_{j\rangle k} + K_{ij}^{GE} + K_{ij}^{GM}, \tag{252}$$

where the gravito-electric (GE) and gravito-magnetic (GM) post-Newtonian tidal matrices read

$$\begin{aligned} K_{ij}^{GE} &= \Phi_{\oplus,\langle ij \rangle} - 2 W_\oplus W_{\oplus,\langle ij \rangle} - \tfrac{1}{2} \ddot{\Psi}_{\oplus,\langle ij \rangle} - 2 V_T^{\langle i} \dot{W}_\oplus^{,j\rangle} + 2 V_T^2 W_{\oplus,\langle ij \rangle} - 3 V_T^k V_T^{\langle i} W_\oplus^{,j\rangle k} + 3 A_T^{\langle i} A_T^{j\rangle} - 6 A_T^{\langle i} W_\oplus^{,j\rangle} - 3 E_{\langle i} E_{j\rangle} \,, \\ K_{ij}^{GM} &= 4 \Big(\dot{W}_\oplus^{\langle i,j \rangle} + V_T^k W_\oplus^{\langle i,j \rangle k} - V_T^k W_\oplus^{k,\langle ij \rangle} \Big). \end{aligned} \tag{253}$$

For the partial derivatives of Earth's gravitational potential one obtains

$$\begin{aligned} W_{\oplus,\langle ij \rangle} &= \frac{3\mu_\oplus}{R^5} \left(X^i X^j - \frac{R^2}{3} \delta^{ij} \right) - 2G \sum_{l=2}^{\infty} \frac{(2l+1)!!}{(l-1)!} \frac{M_{L(i}^\oplus X^{j)} X^L}{R^{2l+3}} + G \sum_{l=2}^{\infty} \frac{(2l+3)!!}{l!} \frac{M_L^\oplus X^L}{R^{2l+5}} \left(X^i X^j - \frac{R^2}{2l+3} \delta^{ij} \right), \\ W_{\oplus,\langle ijk \rangle} &= -\frac{15\mu_\oplus}{R^7} \left(X^i X^j X^k - \frac{R^2}{5} \left(\delta^{ij} X^k + \delta^{jk} X^i + \delta^{ik} X^j \right) \right). \end{aligned} \tag{254}$$

Furthermore, the following Newtonian (point mass) approximations for the tidal terms should be sufficient:

$$Q_{ij} = \sum_{A \neq \oplus} \frac{3\mu_A}{r_{\oplus A}^3} \left(n_{\oplus A}^i n_{\oplus A}^j - \frac{1}{3} \delta^{ij} \right), \qquad Q_{ijk} = \sum_{A \neq \oplus} \frac{15\mu_A}{r_{\oplus A}^4} \left(n_{\oplus A}^i n_{\oplus A}^j n_{\oplus A}^k - \frac{1}{5} \left(\delta^{ij} n_{\oplus A}^k + \delta^{jk} n_{\oplus A}^i + \delta^{ik} n_{\oplus A}^j \right) \right) \tag{255}$$

with $\mathbf{r}_{\oplus A}(t) = \mathbf{x}_\oplus(t) - \mathbf{x}_A(t)$, $r_{\oplus A} = |\mathbf{r}_{\oplus A}|$, and $\mathbf{n}_{\oplus A} = \mathbf{r}_{\oplus A}/r_{\oplus A}$.

Under certain assumptions on Earth's body, e.g., stationary rotation and invariant multipole moments, one can simplify the expressions for the gravito-electric and gravito-magnetic tidal matrices, retaining only major terms:

$$\begin{aligned} K_{ij}^{GE} &= \frac{3GM_\oplus}{R^3}\left(\left(2V^2 - \frac{3GM_\oplus}{R}\right)N^{\langle i}N^{j\rangle} + V^{\langle i}V^{j\rangle} - 3(\mathbf{N}\cdot\mathbf{V})V^{\langle i}N^{j\rangle}\right), \\ K_{ij}^{GM} &= \frac{3GS_\oplus}{R^4}\left(5(\mathbf{N}\cdot\mathbf{V})(\mathbf{s}\times\mathbf{N})^{\langle i}N^{j\rangle} - 5\big((\mathbf{s}\times\mathbf{N})\cdot\mathbf{V}\big)N^{\langle i}N^{j\rangle} - 3(\mathbf{s}\times\mathbf{V})^{\langle i}N^{j\rangle} - (\mathbf{s}\times\mathbf{N})^{\langle i}V^{j\rangle}\right), \end{aligned} \tag{256}$$

where $N^i = \mathbf{N} = \mathbf{X}/R$ represents the GCRS-related unit vector. Kopejkin et al. [299] provide explicit formulas for an Earth orbiting satellite, i.e. the SRS case, using classical Keplerian elements. Following this approach one can draw conclusions on the measurability and separability of individual tidal terms based on a certain orbit configuration. These results are important for future gradiometry space missions.

30.17 Definition of a relativistic geoid

Focussing on height determination by means of atomic clock readings, naturally the question of a suitable reference surface arises. The classical definition of the geoid in Newtonian gravity has to be refined within the framework of relativistic geodesy. This issue was also already addressed in parts by Bjerhammar [46].

Even though traditional geoid computation today aims at the *mm*-level, it still neglects any post-Newtonian corrections that are of the same order of magnitude, which eventually leads to a bias in the geoid determination.

Two seemingly distinct definitions for a relativistic geoid do exist (Soffel [511]):

The first one is based on the rate ratio of two time scales, namely proper time τ versus GCRS time T, the latter being regarded as kind of a coordinate time. For any point of the two-dimensional reference surface (T-geoid) one simply stipulates that $\mathrm{d}\tau/\mathrm{d}T = \text{const} =: W_0$, assuming that τ relates to a stationary observer with respect to the geocentric frame, i.e. $\mathbf{R}_A = \mathbf{0}$ and $\dot{\mathbf{R}}_A = \mathbf{0}$. In order to derive an explicit approximative expression for W_0 remember equations (245) and (247), thus (remark: there was a mismatch in some coefficients between Kopejkin [295] and Kopejkin et al. [299])

$$\begin{aligned} W_0 &= \tfrac{1}{2}V_A^2 + W_\oplus + Q_i X_A^i + \tfrac{1}{2}\,Q_{ij}X_A^iX_A^j + \tfrac{1}{6}\,Q_{ijk}X_A^iX_A^jX_A^k + \\ &+ \frac{1}{c^2}\left(\tfrac{1}{8}\,V_A^4 + \tfrac{3}{2}\,V_A^2W_\oplus - \tfrac{1}{2}\,W_\oplus^2 - 4V_A^aW_\oplus^a + \Phi_\oplus - \tfrac{1}{2}\,\ddot{\Psi}_\oplus\right) \\ &\approx \tfrac{1}{2}V_A^2 + W_\oplus + \frac{1}{c^2}\left(\Phi_\oplus - \tfrac{1}{2}W_\oplus^2\right) = \text{const.} \end{aligned} \tag{257}$$

Alternatively, we could have used equation (246) for the differentiation, of course. Even if $\dot{\mathbf{R}}_A = \mathbf{0}$, and in general $\mathbf{V}_A \neq \mathbf{0}$, the evaluation of the product rule outcome for the second bracket term of the first equation in (245) yields a zero result, i.e. $\mathbf{A}_A\cdot\mathbf{R}_A + \mathbf{V}_A\cdot\dot{\mathbf{R}}_A = \mathbf{0}$. Furthermore, due to the fact that W_0 by definition is a constant anywhere on our surface, we can drop the subindex for the observers' (clock's) geocentric position and velocity altogether.

The second relativistic geoid definition is based on the inertial acceleration (of gravity) which is directly related to the concept of a plumb line. Now, at any point of the two-dimensional reference surface (A-geoid), one stipulates that $T = \text{const}$, thus $\mathrm{d}T = 0$, and the surface being orthogonal to the plumb line, represented by the topocentric direction of Earth's gravity, i.e., the scalar product $\mathbf{E}\cdot\mathrm{d}\boldsymbol{\xi} = E_i\,\mathrm{d}\xi^i = 0$. This time we exploit the second equation of (245) by taking its differential. Again one can suppress the subindex because relation (245) shall hold for any point of the surface. Applying the defining condition, expressing E_i with help of equations (250), taking the differential of W_0 based on equation (257), and assuming a constant rigid-body rotation for the Earth's matter, Kopejkin et al. [299] finally obtain the simple relationship

$$E_i\,\mathrm{d}\xi^i = \mathrm{d}W_0 = 0, \tag{258}$$

which is in agreement with the first definition. Thus, both definitions are equivalent. The gradient of W_0 can be used to express Earth's gravity force on the surface of our defined relativistic geoid:

$$E_i = \left.\frac{\partial W_0}{\partial \xi^i}\right|_{T=\text{const}} = \frac{\partial W_0}{\partial X^k}\frac{\partial X^k}{\partial \xi^i}\,. \tag{259}$$

The question arises whether the relativistic geoid can be regarded as an equilibrium figure. Kopejkin et al. [299] affirm this question by making the assumption that Earth's matter is a rigidly rotating perfect fluid. This result would eventually back the introduction of a relativistic level surface. Nevertheless, this issue has to be addressed carefully again in the course of practical clock experiments, i.e., chronometric leveling campaigns, especially in the context of realistic tidal considerations.

Clock based height determination

As mentioned before, various geophysical effects significantly change the potential regime at a clock site within a comparatively short period of time. One of the most obvious effects is due to the Earth's tides. Additionally, the resulting spatial site displacement itself may lead to a non-negligible frequency shift due to the relativistic Doppler effect. On the other hand, accurate clock readings still require comparatively long interrogation times. In the following we exemplarily estimate and illustrate these tidal effects, which are crucial for the chronometric leveling approach, i.e., the direct determination of potential differences by means of (atomic) clock readings.

The first paragraphs motivate this task a little bit further via a discussion of various aspects of classical (non-relativistic) methods of height determination as traditionally being applied within physical geodesy. Especially, we recap selected time scales and height definitions. A larger portion of this chapter is devoted to a detailed treatment of tides and its implications. At the end we present some estimates on the feasibility, i.e., necessary sensitivity, of proposed clock comparison campaigns.

31 Practical time scales and their relations

Geodesy and related disciplines make use of various time scales. A detailed description is provided by updated IERS conventions (e.g. IERS TN36 [257]). Several links between individual time scales are depicted in figure 1, whereas the essential relations are given by (Müller [377], Nothnagel et al. [394])

$$\begin{aligned}
\mathrm{UT1} &= \mathrm{UT0} - \big(x_p \sin\lambda + y_p \cos\lambda\big)\tan\phi\,,\\
\mathrm{UT2} &= \mathrm{UT1} + 0.022\sin 2\pi T_B - 0.012\cos 2\pi T_B - 0.006\sin 4\pi T_B + 0.007\cos 4\pi T_B\,,\\
\mathrm{UT1R} &= \mathrm{UT1} - \sum_{i=1}^{41} A_i \sin\left(\sum_{j=1}^{5} k_{ij}\alpha_j\right),\\
\mathrm{UT1D} &= \mathrm{UT1} - \sum_{i=1}^{8}\big(D_i\sin\xi_i + E_i\cos\xi_i\big) \quad \text{with} \quad \xi_i = \sum_{j=1}^{6} c_{ij}\gamma_i + \Phi_i,\\
\mathrm{MLT} &= \mathrm{UT1} + \lambda\,,\\
\mathrm{TLT} &= \mathrm{MLT} + \mathrm{EOT}\,,
\end{aligned} \tag{260}$$

for solar time scales, and by

$$\begin{aligned}
\mathrm{GMST} &= \mathrm{UT1} + 24110.54841 + 8640184.812866T + 0.093104T^2 - 0.0000062T^3,\\
\mathrm{LMST} &= \mathrm{GMST} + \lambda\,,\\
\mathrm{GAST} &= \mathrm{GMST} + \Delta\psi\cos\varepsilon_A + 0''.00264\sin\Omega + 0''.000063\sin 2\Omega\,,\\
\mathrm{LAST} &= \mathrm{GAST} + \lambda\,,
\end{aligned} \tag{261}$$

with $T = (T_{JD} - 2451545.0)/36525$ for sidereal time scales, and by

$$\begin{aligned}
\text{UT1-UTC} &= \text{defined by IERS by means of space geodetic techniques}\,,\\
\mathrm{TZT} &= \mathrm{UTC} \pm k(\lambda)\,\mathrm{hrs} \quad \text{(longitude dependent Time Zone Time)}\,,\\
\mathrm{JD} &= \text{several algorithms (depending on the time scale used) exist to calculate JD from civil date}\\
&\quad\ \text{(year, month, day, hours, minutes, seconds, leap year rule), cf. Vallado/McClain [561]}\,,
\end{aligned} \tag{262}$$

for universal time scales, and by

$$\begin{aligned}
\mathrm{TAI} &= \mathrm{UTC} + \begin{cases} \ \vdots & \text{(for a historical list of leap seconds, starting in 1972, cf. NIST [392])} \\ 34\,s & \text{since 01.01.2009} \\ 35\,s & \text{since 01.07.2012} \end{cases},\\
\text{TP-TAI} &= \text{determined by means of astronomical observations}\,,\\
\text{TGPS-TAI} &= \text{comparison of different defining atomic clock ensembles}\,,
\end{aligned} \tag{263}$$

for atomic time scales, and by

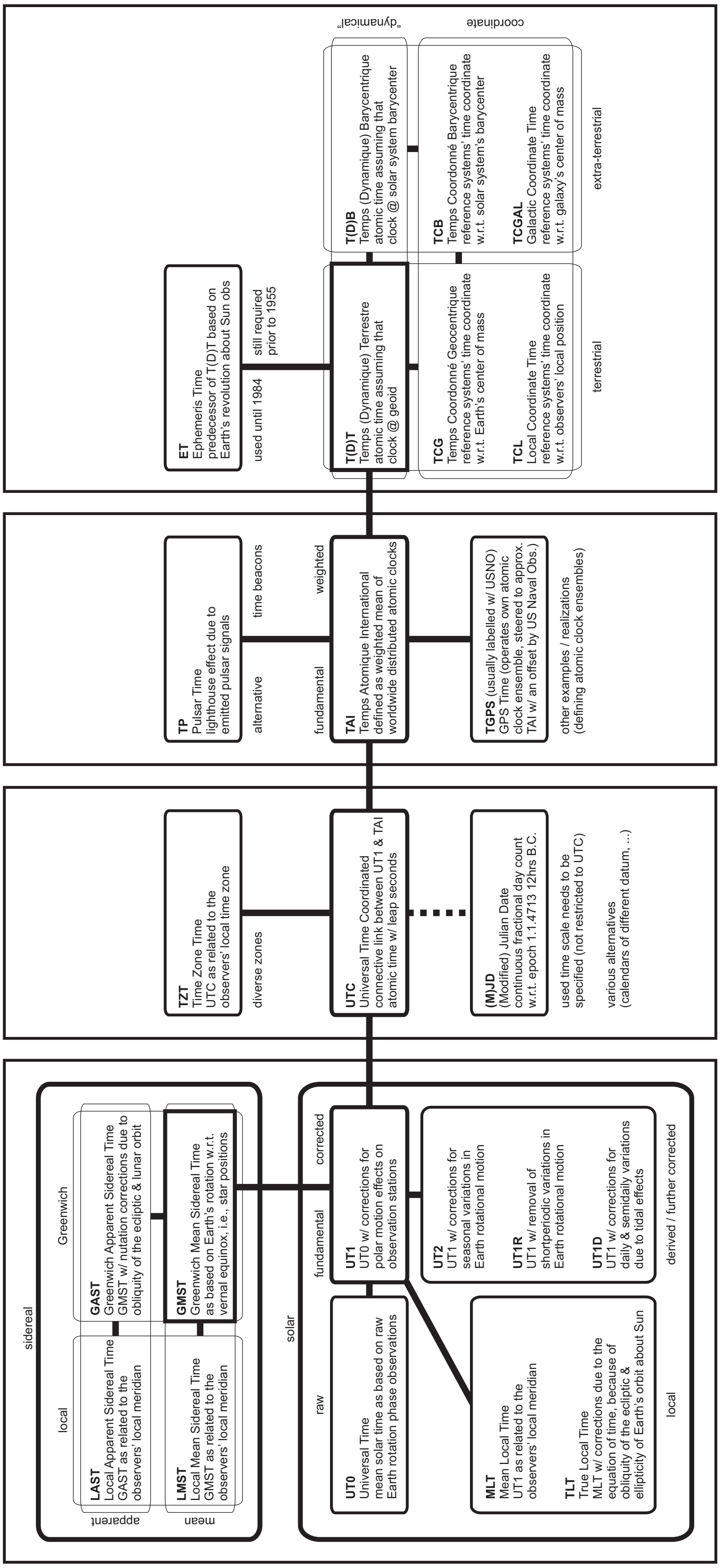

Fig. 1: Time scales and their relations, based on illustrations in Müller [377].

$$
\begin{aligned}
\mathrm{TDT} &= \mathrm{TAI} + 32.184s\,,\\
\mathrm{TT} &\equiv \mathrm{TDT}\,,\\
\mathrm{TDB} &= \mathrm{TDT} + \frac{1}{c^2}\,\mathbf{v}_\oplus\big(\mathbf{x}-\mathbf{x}_\oplus\big) + \sum_{i=1}^{122}\big(B0_i + B1_i T + B2_i T^2\big)\sin(n_i T + \alpha_i),\\
\mathrm{TB} &\equiv \mathrm{TDB}\,,\\
\mathrm{ET} &\approx \mathrm{TDT}\ \text{(Ephemeris Time was not specified based on the theory of relativity;}\\
&\qquad\quad \text{in its uncertainty ET approximates a number of dynamical time scales)}\,,\\
\mathrm{TCG} &= \mathrm{TDT} + \underbrace{\frac{1}{c^2}\,W_\oplus\Big|_{\text{geoid}}}_{=:\,k_\oplus\,\approx\,0.7\cdot 10^{-9}}\ \Delta T_{JD_{1.1.1977}}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{TCB} &= \mathrm{TCG} + \int\!\left(\frac{1}{c^2}\,W_{\text{ext}}\Big|_{\mathbf{x}^B_\oplus} + \frac{1}{c^2}\frac{v^{B\,2}_\oplus}{2}\right)\mathrm{d}t + \frac{1}{c^2}\,\mathbf{v}^B_\oplus\big(\mathbf{x}^B - \mathbf{x}^B_\oplus\big)\\
&= \mathrm{TCG} + \big(\mathrm{TDB}-\mathrm{TDT}\big) + \underbrace{\overline{\left(\frac{1}{c^2}\,W_{\text{ext}}\Big|_{\mathbf{x}^B_\oplus} + \frac{1}{c^2}\frac{v^{B\,2}_\oplus}{2}\right)}}_{=:\,k_B\,\approx\,1.48\cdot 10^{-8}}\Delta T_{JD_{1.1.1977}}\,,\\
&= \mathrm{TDB} + \big(k_\oplus + k_B\big)\Delta T_{JD_{1.1.1977}}
\end{aligned}
\tag{264}
$$

$$
\mathrm{TCGAL} = \mathrm{TCB} + \int\!\left(\frac{1}{c^2}\,W_{\text{ext}}\Big|_{\mathbf{x}^G_B} + \frac{1}{c^2}\frac{v^{G\,2}_B}{2}\right)\mathrm{d}t + \frac{1}{c^2}\,\mathbf{v}^G_B\big(\mathbf{x}^G - \mathbf{x}^G_B\big)
$$

$$
\mathrm{TCL} = \mathrm{TCG} + \int\!\left(\frac{1}{c^2}\,W_{\text{ext}}\Big|_{\mathbf{x}^\oplus_L} + \frac{1}{c^2}\frac{v^{\oplus\,2}_L}{2}\right)\mathrm{d}t + \frac{1}{c^2}\,\mathbf{v}^\oplus_L\big(\mathbf{x}^\oplus - \mathbf{x}^\oplus_L\big)
$$

with $\Delta T_{JD_{1.1.1977}} = \big(T_{JD} - 2443144.5\big)86400s$ for theoretical time scales, where indices $G, B, \oplus$, and L denote the galactic, (solar system) barycentric, Earth's, and local observer's center of mass, respectively. The latter might also refer to a satellite reference system (SRS).

Another common quantity, namely the Earth Rotation Angle (ERA), refers to the Celestial Intermediate Origin (CIO) and represents an angle measured along the intermediate equator of the Celestial Intermediate Pole (CIP) between the CIO and its terrestrial counterpart TIO (Terrestrial Intermediate Origin), the latter being the non-rotating origin of the ITRS. Thus, this quantity plays an important role in the ICRS-ITRS transformation. It is counted positively in retrograde direction and there exists the following linear proportionality to the time scale UT1 (IERS TN29 [255]), the latter of which is affected by the variable speed of the Earth's rotation,

$$
\mathrm{ERA}(T_{UT1}) = 2\pi\big(0.7790572732640 + 1.00273781191135448\,T_{UT1}\big) \tag{265}
$$

with $T_{UT1} = JD_{UT1} - 2451545.0$. Via the so-called „equation of the origins“ (IERS TN36 [257]) it is related to the sidereal time scale GMST by

$$
\mathrm{GMST} = \mathrm{ERA} + 0''\!.014506 + 4612''\!.156534t + 1''\!.3915817t^2 - 0''\!.00000044t^3 - 0''\!.000029956t^4 - 0''\!.0000000368t^5 \tag{266}
$$

with $t = (TT - TT_{\text{01Jan2000 12hrs}})$ in days/36525.

Details on the individual unspecified parameters and terms in equations (260) - (266) are provided by many authors (e.g. Müller [377]). Some of the relations are relevant for relativistic geodesy and were already discussed in the preceding sections, for example in equation (225) which is just an alternative expression for TCB − TCG.

The most relevant time scale transformation, not explicitly shown in figure 1, is the relation between individual atomic clocks' local proper time τ and the geocentric coordinate time TCG as given in equations (245) - (249).

32 Potential differences and classical height systems

The term „height“ (Meyer et al. [360]) can be defined as the metric length of a straight or curvilinear ray of projection of a point onto a reference line (2D case) or reference surface (3D case) (Lelgemann/Petrovic [324]). Following this definition, it is always a geometrical entity and shall be representable in a graphic way.

Different physical scalar fields can be used to set up a specific concept of height. Essentially, one regards the fields' isoscalar surfaces as surfaces of constant height. Exemplarily, gravimetric height determination is based on the gravity field of Earth's masses, barometric height determination relates to the pressure field of Earth's atmosphere, and chronometric height determination will make use of a suitable isolated systems' metric tensor. Here we briefly discuss selected aspects of classical height measurement based on Earth's gravity potential which are of interest for later comparison to the relativistic chronometric leveling approach.

Only coincidentally isoscalar surfaces will be parallel and with a constant metric distance to each other. Thus, one can not assign a unique metric height measure to points on different equipotential surfaces, and any pure geometrical leveling between two given points will yield a path dependent metric height difference. Aside from certain special cases (e.g. radial symmetric equipotential surfaces), the determination of unique (geo)potential differences (or height differences) requires simultaneous measurements of gravity along the leveling path. Basically (Heiskanen/Moritz [239], Torge [550]),

$$W_B - W_A = \int_A^B \mathrm{d}W = \int_A^B \nabla W \cdot \mathrm{d}\mathbf{r} = \int_A^B \mathbf{g} \cdot \mathrm{d}\mathbf{r} = \int_A^B \mathbf{g} \cdot \mathbf{n}\, \mathrm{d}r = -\int_A^B g\, \mathrm{d}n, \tag{267}$$

where we stipulate that equipotential values of the gravity field decrease with increasing height.

In classical physical geodesy a variety of height systems and corresponding reference surfaces exists. Geodesists distinguish geometrically defined heights (e.g. ellipsoidal) from physically defined (metrical) heights (e.g., dynamical, orthometric, spheroidal or normal-orthometric, normal), where the latter are somehow connected to gravity and are derived from raw path dependent heights which do not comprise any consideration of gravity at all. In a strict sense, normal-orthometric heights are rather geometrically defined than physically, as we will point out later. Physically defined height systems can be related to a specific geometrical reference surface, e.g., the level ellipsoid of revolution. For some height transformations this requires an additional transition between geometrical surfaces, too.

Several height definitions theoretically lead to path independency but in practice, due to systematic errors, there will almost always remain a (residual) path dependency. This question is of importance if one has to transform between different height systems. As long as the involved height definitions are path independent, we can use areal transformations, otherwise line-wise transformations apply.

Physically defined heights are related to metric distances along plumb lines, i.e., orthogonal trajectories of the geoid or quasi-geoid, whereas geometrically defined heights are measured along normal lines to a given geometric reference surface for position determination. The latter can be irregular in shape but preponderantly one simply uses ellipsoids of revolution.

Figure 2 depicts various height systems and its relations. For example, orthometric heights are based on the geoid concept, whereas normal heights are related to the quasi-geoid. As a result, its transformation is identical to the difference function between those two reference surfaces.

The geoid (e.g. Heiskanen/Moritz [239], Hofmann-Wellenhof/Moritz [249]) acts as both a physical and geometrical reference surface. A physical reference surface in the sense of an equipotential surface of the vertical datum has the advantage of always being unique, whereas a geometrical reference surface may not always be uniquely defined, e.g., in case of dynamical or normal heights. Constituting a drawback of physical heights, the underlying scalar field may change in time leaving the geometry of pegged height reference stations invariant.

Practically, the geoid can be determined as solution of a free boundary value problem in the sense of Stokes or from the knowledge of Earth's exterior gravity field (as derived by means of satellite geodesy techniques) in combination with a downward continuation. Traditionally, the gravity field and its functionals like (quasi-) geoid heights, deflections of the vertical etc., can be expressed by means of series expansions using spherical harmonics.

One treats the classical geoid as a selected equipotential surface that, after some slight idealization, more or less fits well to the shape of Earth's mean oceanic surface, assuming an equilibrium state for the oceans and atmosphere, i.e., being at rest relative to the rotating Earth. We usually assign the so-called geoidal geopotential value W_0 to this particular equipotential surface. Along with astronomical position coordinates Φ (latitude) and Λ (longitude) one can introduce the height coordinate W in order to define a set of natural coordinates for each point in the exterior space of Earth. Instead of W itself one often uses the corresponding geopotential number $C = W_0 - W$ as height coordinate. Thus (Torge/Müller [551]),

$$C_P = \int_{P_0}^{P} g\, \mathrm{d}n = W_0 - W_P, \tag{268}$$

such that $C_{P_0} = 0$ and $C_B - C_A = W_A - W_B$, where the geopotential number can be expressed in geopotential units (gpu) with $1gpu = 1kgal\, m$. With $g \approx 0.981kgal$ this implies that $1gpu$ approximately corresponds to a

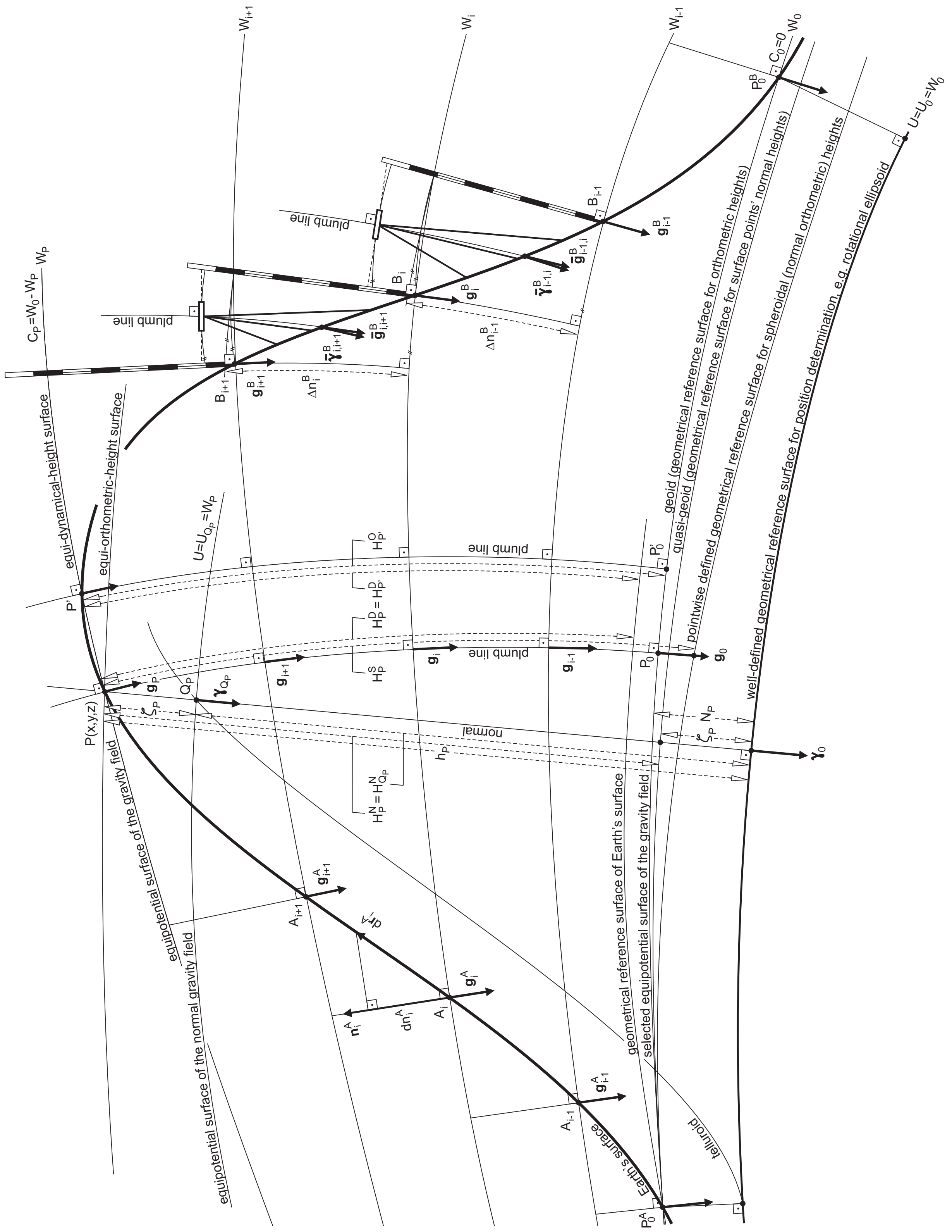

Fig. 2: Classical height reference systems, based on illustrations in Ilk [260]. For details see main text.

metric height difference of $1m$. Geopotential numbers are unique and well-defined but for practical purposes we prefer metric height systems that at all points have the same scale. As a compromise one theoretically defines physical metric heights H by relating geopotential numbers C to certain gravity values via

$$\text{height value} = \frac{\text{geopotential number}}{\text{gravity value}},$$

where the resulting height system mainly depends on the actual choice of the gravity value. In the past there was limited access to precise gravity measurements. Furthermore, practical applications impose different requirements on the accuracy of gravity information. Several alternatives have been proposed which will be denoted by a corresponding upper index: Dynamic, Orthometric, Spheroidal or normal-orthometric, Normal.

In case of spheroidal heights, for the determination of the geopotential number, one replaces real measurements of the gravity value g by calculated normal gravity values γ along the leveling path, based on a resulting normal gravity potential as induced by a chosen reference body. According to equation (268) one gets

$$C_P^D = C_P^O = C_P^N = C_P = \int_{P_0}^{P} g \, \mathrm{d}n \quad \text{and} \quad C_P^S = \int_{P_0}^{P} \gamma \, \mathrm{d}n \,. \tag{269}$$

Dynamic heights make use of a constant (arbitrary) normal gravity value γ_0, e.g., at the geometrical reference surface for the normal gravity potential. If we choose a rotational ellipsoid as reference body for the normal gravity potential then γ_0 will be latitude dependent only. Any equipotential surface of the gravity field is identical to a surface of constant dynamical height. To define a certain geometrical reference for the surface of the Earth, i.e., a dynamical vertical datum, one can simply stake off the same metrical distance (dynamical height value) along the plumb line through any point of a chosen corresponding equipotential surface.

Orthometric heights require the knowledge of a mean gravity value $\bar{g}$ along the path of the plumb line from the geoid point up to the surface point of interest. In general, this information can not be gathered by direct measurements. One rather relies on indirect, e.g. seismic, measurements and/or on assumptions on Earth's interior, i.e., its mass/density distribution, and on its rotational behavior. There exist several slightly differing methods on how to determine $\bar{g}$ in practice.

In order to circumvent some difficulties, one may instead use a mean normal gravity value $\bar{\gamma}$ along the plumb line of the normal gravity field from the geometrical reference surface for the normal gravity potential (e.g. reference ellipsoid of revolution) up to the corresponding telluroid point of our point of interest. The shape of the telluroid, which is defined as the surface for which $W_P = U_{Q_P}$ (cf. figure 2) holds for every point, resembles that of the physical surface of the Earth (Torge [550]). The telluroid itself is not a level surface of the normal gravity field. As a remark, the before mentioned plumb line is not identical to the normal (of the ellipsoid) but in practice its lengths are almost the same. Equipotential surfaces of an ellipsoid of revolution are not ellipsoidal in shape.

Having introduced the concept of geopotential numbers, in a second independent step one defines corresponding metric heights by

$$H_P^D = \frac{C_P}{\gamma_0} \,, \quad H_P^O = \frac{C_P}{\bar{g}} \,, \quad H_P^S = \frac{C_P^S}{\bar{\gamma}} \,, \quad H_P^N = \frac{C_P}{\bar{\gamma}} \,. \tag{270}$$

The precise definition of heights remains a subtle task. There exist arguments against the above mentioned classical view. Normal-orthometric heights have to be regarded as geometrically rather than physically defined, because they are not based on real geopotential differences but on the arbitrary introduction of a normal gravity field. Furthermore, normal heights are based on a pure formal definition. Many artificial concepts, e.g., the quasi-geoid and telluroid, may be expendable and some of the traditional height definitions, e.g. dynamical and normal heights, are only substitutes or auxiliary concepts for more relevant and better interpretable quantities like geopotential numbers, normal-orthometric heights, and ellipsoidal heights (Lelgemann/Petrovic [324]).

Several height measurement methods are available, differing in accuracy, effort for the measurement itself, and post-processing. Many leveling techniques do exist already: geometrical (horizonal line of sight apart from refraction effects), trigonometrical (slant line of sight), geodetic (combination of geometrical leveling plus point-wise gravity measurements along the leveling path), geometrical-astronomical (with zenith angle measurements), trigonometric-astronomical, GNSS (aiming at ellipsoidal heights employing satellite geodesy methods). Chronometric leveling (based on clock comparisons) will become another alternative. Heights may also be determined based on trigonometrical, photogrammetrical, inertial, or barometrical measurement techniques, etc.

The precise raw geometrical leveling result

$$\Delta n_{AB} = \sum_{A}^{B} \Delta n_i \tag{271}$$

has to be supplemented by a reduction R_{AB} due to gravity effects in order to obtain path independent height differences, i.e. $\Delta H_{AB} = \Delta n_{AB} + R_{AB}$. Alternatively, the reductions can directly be applied to corresponding geopotential number differences. The actual reduction itself depends on the requested kind of height.

The following resulting formulas can be applied in practice:

$$
\begin{aligned}
R_{AB}^{D} &= \sum_{A}^{B} \frac{\bar{g}_{i-1,i} - \gamma_0}{\gamma_0}\, \Delta n_i\,, \\
R_{AB}^{O} &= \sum_{A}^{B} \left(\frac{\bar{g}_{i-1,i} - \gamma_0}{\gamma_0}\, \Delta n_i + \frac{\bar{g}_{i-1} - \gamma_0}{\gamma_0}\, H_{i-1}^{O} - \frac{\bar{g}_{i} - \gamma_0}{\gamma_0}\, H_{i}^{O} \right), \\
R_{AB}^{S} &= \sum_{A}^{B} \left(\frac{\bar{\gamma}_{i-1,i} - \gamma_0}{\gamma_0}\, \Delta n_i + \frac{\bar{\gamma}_{i-1} - \gamma_0}{\gamma_0}\, H_{i-1}^{S} - \frac{\bar{\gamma}_{i} - \gamma_0}{\gamma_0}\, H_{i}^{S} \right), \\
R_{AB}^{N} &= \sum_{A}^{B} \left(\frac{\bar{g}_{i-1,i} - \gamma_0}{\gamma_0}\, \Delta n_i + \frac{\bar{\gamma}_{i-1} - \gamma_0}{\gamma_0}\, H_{i-1}^{N} - \frac{\bar{\gamma}_{i} - \gamma_0}{\gamma_0}\, H_{i}^{N} \right).
\end{aligned}
\tag{272}
$$

Obviously,

$$
R_{AB}^{N} = R_{AB}^{S} + \sum_{A}^{B} \frac{\bar{g}_{i-1,i} - \bar{\gamma}_{i-1,i}}{\gamma_0}\, \Delta n_{i-1,i}\,. \tag{273}
$$

Normal-orthometric heights will differ noticeably from normal heights only in case of very inhomogeneous gravitational fields and significant height differences. In equation (272) dashed quantities indicate mean values, where those with a single index are averaged along the plumb line and those with two indices are averaged between two neighboring surface points along the leveling path.

33 The global vertical datum problem

Besides the task of height system definition, the linking problem between regional vertical datum systems has to be addressed. One finally aims at an integrated vertical datum system which possibly comprises multiple datum shifts in order to obtain a consistent global vertical datum system (Ihde/Sánchez [259]).

Based on the availability of a precise high-resolution global gravity field model, one traditionally formulates this task as a geodetic scalar boundary value problem which is related to the determination of a mean sea level (Rummel/Teunissen [462], Rapp [445], Sánchez [467]). Neighboring height systems are often separated by marine areas, either on local scale within insular/coastal regions or on a global scale between individual continents. Even in continental Europe there are various national height systems in use that show significant discrepancies. Several applications in Earth system research require unified height systems on a continental and global scale.

The reference surfaces of regional vertical datum systems can be regarded as equipotential surface sections belonging to one and the same gravity field but, in general, referring to different levels, see figure 3. Regionally defined geoid sections will have no direct relation to each other. Instead, on a global scale, we face numerous datum shifts that have to be determined precisely. The gravity field model must be of a very high (spatial) resolution because of its usual representation in spherical harmonics. As a remark, alternative (locally valid) representations do exist, e.g., radial base functions. Any geoidal height or regional gravity measurement at an arbitrary location, in principle, depends on globally distributed gravity anomalies which should be related to a (still non-realized) unified and consistent reference surface.

Theoretically, the equipotential surface $W_0 = \text{const}$ could act as a global vertical datum, but existing global geoid models are by far not precise enough (in terms of resolution) in order to allow for a sufficient solution. Details on the practicability of W_0 and its approximative value of $W_0 = 62636856.0\, m^2/s^2$ for the realization of a global vertical reference system (GVRS) are provided in Burša et al. [79].

Today, geodesists mostly apply global space geodetic techniques (delivering ellipsoidal heights), supplemented by regional leveling and terrestrial gravity measurements combined with satellite gravimetry, in order to tackle the vertical datum integration problem. In continental areas, GNSS, SLR, and VLBI are applied, whereas oceanic areas are mainly covered by altimetry. Traditionally, tide gauges are being utilized in coastal regions to realize the linking of continental and marine topographies. Several solution strategies evolved in the past. Combination of satellite positioning with geodetic leveling can be applied in continental areas. Oceanic areas allow for a combination of satellite altimetry with oceanic leveling (steric and/or dynamic/geostrophic) (e.g. Stewart [528]). Fischer [186] discusses conceptual differences between genuine geodetic and oceanographic measurements and reference surfaces. The collection of precise data with high spatial and temporal resolution, especially in the marine areas, is greatly facilitated by dedicated space missions like GRACE or GOCE (Rummel et al. [463]).

The oceanic leveling technique in analogy with geodetic leveling is being used to derive geopotential differences and dynamic height differences between different ocean surface points by means of a line-wise integration of

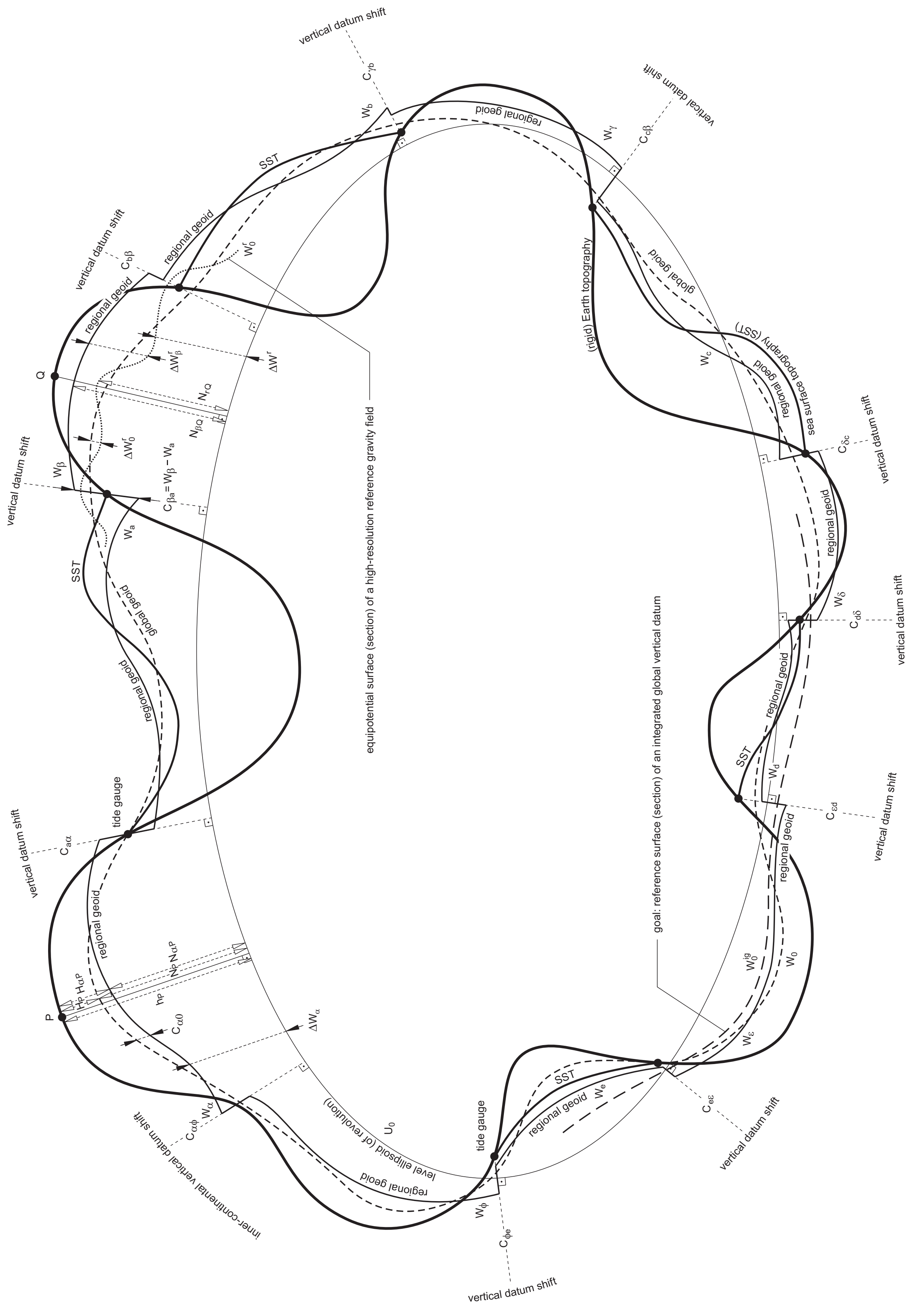

Fig. 3: Integrated global vertical datum problem, based on illustrations in Ilk [260]. For details see main text.

geometric height increments in combination with gravity information. But this time it depends on a different set of observations, namely surface and volume forces which can be modeled by such parameters as temperature, pressure, saltiness, and so on. Furthermore, the technique is based on the hydrodynamic equation of motion of the individual water particles.

In comparison we have (e.g. Cartwright [93])

$$\begin{aligned} &\text{geodetic leveling:} \quad W_{AB} = W_B - W_A = \int_A^B \mathbf{g} \cdot \mathrm{d}\mathbf{r}\,, \\ &\text{oceanic leveling:} \quad W_{AB} = W_B - W_A = \int_A^B \left(\frac{1}{\rho} \nabla p + 2\boldsymbol{\omega}_\oplus \times \mathbf{v} \right) \cdot \mathrm{d}\mathbf{r}\,, \end{aligned} \tag{274}$$

where ρ denotes the ocean water density, p is a pressure function, and $\mathbf{v}$ represents the velocity of the water particles. Depending on the integration path there are several ways to evaluate the integral. For example, the steric leveling technique requires depth dependent density measurements by applying

$$W_{AB} = \int_{B'}^{B} \frac{1}{\rho}\,\mathrm{d}p - \int_{A'}^{A} \frac{1}{\rho}\,\mathrm{d}p\,, \tag{275}$$

with A' and B' being submarine points along the corresponding plumb lines through A and B which are located on the same isobaric surface (level of no-motion), i.e., $p_{A'} = p_{B'}$ which is equivalent to $W_{A'} = W_{B'}$. Other ways comprise different observations, e.g., surface water velocity measurements in case of the dynamic/geostrophic leveling technique.

Regarding the problem of an integrated global vertical datum system, there already exist classical theoretical approaches, but a satisfying solution is still missing. One major limiting factor, in practice, is the insufficient amount of accurate and globally distributed data. Furthermore, the dynamical ocean topography can not be modeled with the same level of precision as associated with today's regional height systems. Additionally, the various existing height systems are not consistent in definition and level of accuracy. These drawbacks prevent us from bridging oceanic areas and linking of the individual systems. Significant progress in the introduction of a globally valid vertical datum is expected by the successful revival of an alternative approach, namely the chronometric leveling by making use of precise atomic clocks.

In order to predict and study the usability of proposed clock experiment setups for the precise chronometric determination of potential differences, we will start with simple simulations on the expected time-varying state (due to tides) of an ensemble of clocks purely in terms of relative motion to each other and its individual positions within the same gravitational field. For any calculations, a specific geographical distribution of clock sites will be chosen, i.e., we consider this (geophysical) state information as accurately known and not as possibly unknown parameters to be additionally estimated for by analysis of the clock readings in a subsequent adjustment procedure. This idealized situation enables us to assess the influence of individual effects on planned observations. Especially, we can separate major geophysical phenomena from instrumentarium specific performance issues that finally enter our judgement on its future geodetic usability.

The precise determination of heights within the framework of relativistic geodesy requires the usage of a consistently defined set of reference systems and its realizations, i.e. its materialization by corresponding reference frames. Several initiatives exist among geodesists (e.g. Nothnagel et al. [394]) to ensure the implementation of the latest refinements in modeling and resulting recommendations on the various definitions and conventions, and to achieve further improvements in terms of precision, accuracy, and long-term stability. Some fundamental remarks on the coherent treatment of concepts like reference systems, reference frames, geodetic datum and its subtleties are addressed in Drewes [145].

Driven by the demands of other communities' requirements (e.g. in climate or environmental research), one currently aims at globally valid four-dimensional reference systems (Soffel/Langhans [514]) (geometric and dynamic) that provide benchmarks to which one can reliably refer, over decades, Earth system changes on the millimeter level maintaining a stability of less than a millimeter per year. This in turn translates to necessary measurements of time with a sufficient high level of accuracy that could be achieved by applying upcoming optical clocks and novel time comparison techniques.

34 Introductory remarks on the displacement of observation sites

The displacement of reference points, that are supposed to be stationary, is a general problem which applies to every measurement technique. Additionally, especially when using a body fixed reference frame, one has to be aware of the fact that one can relate any site coordinates either to the body's center of mass (COM) or to its center of figure (COF) which are, in general, not identical (Blewitt [57]). The term reference point here refers either to the precise location of the starting point and/or endpoint of an observation, e.g., the phase center of a VLBI antenna, or the actual position of the interrogated atoms within the apparatus of an optical clock, or to a physical marker (especially useful in simulations) representing the location of the whole instrument and being rigidly attached to the body, e.g., to Earth's solid crust. In any case, we distinguish between several sources of displacement, namely geophysical and non-geophysical effects.

Most prominent are tidal and non-tidal loading effects as well as instrument specific effects acting upon its internal reference point, e.g., by thermally induced deformation of its apparatus (IERS TN36 [257]). The latter are non-geophysical effects causing also position-like displacements. They can be treated by a technique/instrument-dependent introduction of reference models, e.g., for temperature. Solid Earth deformations are due to (ocean plus atmospheric) tidal loading and body tides, caused by external tide-generating potentials, Earth rotation variations, and ocean pole tide loading. The total tide is a combination of load tides and body tides (Agnew [2]). Harrison [233] collects benchmark papers on various tidal aspects, e.g., tidal forces and deformations, as well as tidal tilt and strain issues. In a terrestrial coordinate system one could express the total tidal displacement $\boldsymbol{\Delta}$ of an observers' (atomic clock) location $\mathbf{r}_A$ as a sum $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\text{sol}} + \boldsymbol{\Delta}_{\text{pol}} + \boldsymbol{\Delta}_{\text{ocn}} + \boldsymbol{\Delta}_{\text{atm}} + \cdots$ (Sovers et al. [515]).

Non-tidal loading effects due to atmosphere, continental water or post-glacial rebound are superimposed by post-seismic motions, monument stability, and other effects of higher order. Geophysical displacement effects, in general, are driven by mass redistributions in the system Earth and can be of seasonal, secular, or episodic character. Details on the periodicity and magnitude (mainly on the *cm*-level) of individual tidal and non-tidal loading effects are provided by Nothnagel et al. [394]. This issue will be picked up again in a later paragraph.

As far as crustal markers for precise terrestrial reference frames are concerned, one has to consider not only simple large-scale linear velocity models, as motivated by the theory of plate tectonics, but also non-linear motion of the sites due to co-seismic and post-seismic displacements, transient and loading phenomena, local instabilities, groundwater variations, and so on. With growing demands on the reference frames' accuracy the overall picture gets more and more complex. In addition to a physically meaningful and consistent parametrization of the atmosphere, oceans, hydrosphere, and cryosphere, it requires suitable mathematical approaches (e.g., theoretical functions versus regression between physical signals and geodetic parameters) in order to model the site kinematics correctly. In a first approximation one had to estimate low-degree loading coefficients, at least.

In general, modeling errors exceed the precision of observations, which is also true for today's atomic clock readings. This opens a new field of applications for optical clock ensembles, namely the near-realtime determination and availability of reference frames, e.g., a worldwide unified height system, which comprises the detection of the reference sites' displacements as fast as possible. Currently, this could be achieved mainly by so-called GNSS seismology based on receivers that operate on observation frequencies between about 50 and 100 Hz.

35 Introductory remarks on tides and the tidal potential

The simulation of (idealized) clock states involves the consideration of tides, i.e., the knowledge of the tide-generating (or external) potential and its combined effect at the clock sites. Setting up a resulting formula due to superimposed gravitational action of several third bodies in the neighborhood of Earth requires access to a suitable solar-system ephemeris. Currently, there are three major renowned ephemerides available, namely DE421 (Folkner et al. [193]), INPOP10b (Fienga et al. [183], [184], Manche [341]), and EPM2008 (Pitjeva [423]). Another sets of ephemerides are under development (e.g. Nothnagel [394]). In contrast to analytical orbit integration approaches based on the Newtonian theory of gravitation in the days of Brown [70], [71] or Newcomb [388], today's ephemerides, with only a few exceptions, result from numerical integrations of post-Newtonian equations of motion, i.e., the relativistic Einstein-Infeld-Hoffman equations of motion (Brumberg [75]).

Any tide predicting software package will make use of relevant ephemeris data, either directly or in a simplified version via tailored tide models comprising only a selection of significant tidal constituents. Newton's theory of gravitation clearly constituted a starting point in the modern treatment of tides (Cartwright [93]), even though the tide problem itself was not occasioning this new ideas. Tidal issues are treated in his Principia mainly in books I (starting in proposition 66 with the variation of the Moon due to solar perturbation) and III (discussing the consequences beginning with proposition 24, and especially in propositions 36 and 37 on the quantification of tidal forces), and again in the Principia's supplement („The System of the World“). To some extent, his arguments and results are not consistent throughout the Principia.

Before Newton, many other great scientists tried to give plausible explanations of the various observed tidal phenomena, e.g. Galileo Galilei (Aiton/Burstyn [4], Brown [72], Clutton-Brock/Topper [104]). The work of Laplace on the ocean's response to the tidal forces marks just another cornerstone in the long history of tidal theories (Cartwright [93]), paving the way for the fields of hydrodynamics and geophysical fluid dynamics in general. Instead of assuming a quasi-static response (as Newton did), Laplace set up a system of equations, later labeled Laplace's Tidal Equations (LTE), that describe the response by dynamic relations.

In principle, tides can be computed in various ways (Agnew [2]). In a direct manner one would simply use the ephemerides to compute Earth-fixed geographical coordinates of the sub-body points and corresponding distances. This allows for a point-wise calculation of the tidal potential W_{tidal}, either for a specific location or as a distribution over the whole Earth. Furthermore, any other observable, e.g. tidal tilt or strain, can be derived without any need to perform the initial calculations from celestial mechanics again. On the other hand, the final accuracy of the computation solely depends on the accuracy of the ephemeris and there is a strong coupling between the astronomical and geophysical part.

For example, angular errors for the sub-body point of a few arcseconds in combination with an error in the normalized, i.e. relative, change of distance for the third bodies on the order of 10^{-5} would result in a relative error of 10^{-4} for W_{tidal}/g. These figures already usually exceed the accuracy of classical instrumentation for tide measurements. Remark: the assumed angular errors are equivalent to an error of a few hundred meters in the location of the sub-body point and this in turn, due to the rotation of the Earth, relates to an accuracy requirement for the timing of the data on the order of about a second (Agnew [2]).

Another common way of computing tides is the harmonic expansion of the tidal potential using a sum of sinusoids (tidal harmonics) where the amplitudes depend on the sub-body point's geographical parameters and the frequencies and phases are related to combinations of the third bodies' astronomical parameters. In doing so one decouples the tidal potential from astronomical considerations. Harmonic amplitudes and frequencies can thus be valid for a longer time span. Working in the spectral domain has the advantage of applying the same frequencies for various tidal phenomena, given a linear response in the driving potential. The complete separation of time-varying astronomical quantities from station coordinates, first achieved by Cartwright [93], is thus of great benefit in the actual computation of different kinds of tides, cf. the editor's note in Harrison [233] following the benchmark paper of Bartels [34].

Within the geophysical/oceanographic community, the once challenging problem of tides, especially the behavior of tides in the deep ocean, is sometimes regarded as being nearly finally solved (Cartwright [93]). The various aspects of tides, i.e. its theory, physical observation, (harmonic) analysis of data, or prediction techniques, were studied in great detail in the past and had reached a level of precision and accuracy that seemed to be sufficient for most applications (e.g. Schureman [489], Godin [216], Melchior [357], Gill [208], Marchuk/Kagan [345], Pugh [435]). In future, tidal research will most likely focus on specialized applications of tidal theory, e.g., internal motions, mixing on density layers, air-sea interaction, Earth-tide dissipation, and so on (Cartwright [93]). The study of variable Earth rotation at tidal frequencies, and improved energy dissipation estimates are of geodetic importance, e.g., in research on Earth-Moon dynamics.

36 The modeling of tides

With the ongoing enormous progress in measurement technology, a growing number of geophysical effects on different scales must be taken into account to model the tides for a subsequent analysis of data. Accurate tide modeling is important not only for chronometric leveling but for many different tasks, e.g., dedicated gravity field space missions (Wünsch et al. [605]), detection of gravitational waves (Raab/Fine [441]), or navigation of deep space probes (Moyer [372]).

Starting with the first serious developments of the tide-generating potential due to the Moon and the Sun (Doodson [143]) much progress has been made to include more and more tidal constituents (Cartwright/Tayler [91], Cartwrigth/Edden [92], Büllesfeld [76], Tamura [538], Hartmann/Wenzel [234]), incorporating additional solar-system bodies like the major planets. The harmonic development now contains several thousand individual waves with more than 1000 of them due to the direct effect of third bodies. One of the latest refinements even comprises some 27000 terms in total, and it applies a Poisson series expansion instead of the classical Fourier analysis. This finally results in a maximum error for the accuracy in the calculation of gravity tides at mid-latitude stations of less than one nanogal over a time span of 600 years centered at year 1900 (Kudryavtsev [303]). All these derivations are still being carried out in Newtonian approximation.

Basically, the harmonic decomposition allows for the identification of the most significant tidal constituents and its physical cause. As with the classical derivation of the gravitational potential itself, one can distinguish between individual terms of different degree n and order m. For instance, the $(n, m) = (2, 1)$-term represents diurnal degree-2 tides which itself contains three single harmonics differing in its arguments and amplitudes. Accordingly, one decomposes higher degree and order terms. Depending on the celestial/third bodies taken into account, one can classify the tidal constituents with respect to its arguments into long-period, diurnal, semidiurnal, etc. tides. In a simplified case one only considers the Moon and the Sun as tide-generating third bodies and introduces various associated cyclic orbital motions, such that the frequency f_k (or phase ϕ_k) of each harmonic can be written as a sum of multiples of a few fundamental (astronomical) frequencies (or phases). So, for any (n,m)-term one gets an alternative expression for its argument as (Agnew [2])

$$2\pi f_k t + \phi_k = 2\pi \left(\sum_{l=1}^{6} D_{lk} f_l \right) t + \sum_{l=1}^{6} D_{lk} \varphi_l \,, \tag{276}$$

where f_l are the basic astronomical/tidal frequencies and φ_l represents corresponding phases of these at a suitable epoch, and D_{lk} corresponds to the so-called Cartwright-Tayler codes. This derivation traces back to the algebraic treatment of Doodson [143] based on an analytical ephemeris, comprising the following six astronomical cycles with periodicities from hours to years: lunar day, tropical month (Moon's longitude), solar year (Sun's longitude), lunar perigee, lunar node, and solar perigee. Remark: 'longitude' here refers to celestial longitude measured along the ecliptic. A shorthand notation, namely the six-digit Doodson number, was introduced to illustrate the dependency of the argument on the individual cycles. This compact form also facilitates further sub-classification of the tidal constituents as being applied in tidal spectroscopy (Munk/Cartwright [383]).

For a first simple simulation of a clock's geophysical state, we assume a single earthbound clock, located on the solid/crustal surface of the Earth. Solid Earth tides are easier to model than ocean tides because the medium is more rigid and the geometry of the problem is easier to handle (Agnew [2]). Roughly speaking, tidal forces deform solid Earth by a few decimeters over a time span of a few hours, depending on the latitude. In addition, ocean tides create a periodic load on Earth's surface, giving rise to further displacements of a few centimeters. Ocean tides cause a motion of solid Earth's COM due to motion of the oceans' COM.

In general, current tidal models do not account for the non-conservation (dissipation) of the Earth's total angular momentum via interchange with solar system bodies, especially due to the Moon and the Sun (Sovers et al. [515]). Atmospheric tides also lead to direct and indirect effects that might not be negligible (Gill [208], Ponte/Ray [425]). Other effects are significant more on the millimeter level, e.g., direct contributions of the planets to solid Earth tidal displacements, or resonances with the Earth's free-core nutation. Potential interactions of this kind have to be revised in a consistent and fully-relativistic manner, in future.

37 Tidal displacement and the role of Love and Shida numbers

Changing gravitational attraction shifts the equipotential surfaces of the gravity field of the Earth. This causes tidal forces in its interior that deform the physical surface. The solid Earth resists this force, therefore the deformation is damped. In an idealized case (the Earth is assumed to be oceanless, spherical, non-rotating, isotropic, and elastic with elastic properties varying only with depth), the crustal deformation, i.e. solid Earth's tidal response (assumed to be an equilibrium one, i.e., a quasi-static theory is applied), is completely specified by a few dimensionless parameters k_n, h_n, and l_n of degree n. They range from 0 (perfect rigidity) to 1 (perfect elasticity). Theoretically, due to anelasticity, one can incorporate the concept of a lag angle in order to account for a time-shifted peak response (Sovers et al. [515]).

The so-called Love (k_n, h_n) and Shida (l_n) numbers quantify the tidally induced additional gravitational potential (k_n), and the resulting radial (h_n) and horizontal (l_n) displacement, respectively. As an example, the Earth model PREM provides the following nominal values for degree 2: $k_2 = 0.2980$, $h_2 = 0.6032$, and $l_2 = 0.0839$ (Agnew [2]). An error of 0.1 in h_2 can cause a displacement error of up to a few centimeters which can be directly inferred from expressions for the tidal (northward, eastward, radial) displacement $\mathbf{d}^{(n)}_{\mathrm{tidal}} = (d^{(n)}_N, d^{(n)}_E, d^{(n)}_R)^T$ in a local system (Špičáková [517]) due to an isolated degree-n term $W^{(n)}_{\mathrm{tidal}}$ of the tide-generating potential:

$$d^{(n)}_N = \frac{l_n}{g}\frac{\partial W^{(n)}_{\mathrm{tidal}}}{\partial \phi_A}\,, \quad d^{(n)}_E = \frac{l_n}{g}\frac{\partial W^{(n)}_{\mathrm{tidal}}}{\partial \lambda_A}\frac{1}{\cos\phi_A}\,, \quad d^{(n)}_R = \frac{h_n}{g} W^{(n)}_{\mathrm{tidal}} \qquad \Rightarrow \qquad \mathbf{d}_{\mathrm{tidal}} = \sum_n \begin{pmatrix} d^{(n)}_N \\ d^{(n)}_E \\ d^{(n)}_R \end{pmatrix}, \tag{277}$$

where the site location is specified by geographical coordinates (ϕ_A, λ_A) and r_A, and it is sufficient to apply the approximation $g = \mu_\oplus / r_A^2$. Remark: because the Earth's figure in reality deviates from a perfect sphere, for highly precise calculations one had to distinguish between radial and vertical displacements, as well as between horizontal and tangential ones.

The actual deviation of alternative/older Earth models with respect to the above stated Love numbers is at least one order of magnitude less than 0.1. Thus, the relative errors in deformation computation will be on the sub-*cm*-level and are therefore negligible with respect to expected measurement accuracies in optical clock experiments. Furthermore, degree-2 (quadrupole) displacements are on the order of a few decimeters, whereas degree-3 (octupole) displacements are limited to a few millimeters only (Sovers et al. [515]). For our purposes we can retain only the quadrupole terms of the tide generating potential, i.e. $\mathbf{d}_{\text{tidal}} \approx \mathbf{d}^{(2)}_{\text{tidal}}$, and thus drop the lower index of the Love numbers (subindex n), i.e., use $h := h_2 \approx 0.60$ and $k := k_2 \approx 0.30$ in the following.

Before the advent of space-geodetic techniques tidal measurements were done by means of tiltmeters and/or tidal gravimeters (Cartwright [93]). The former focused on the determination of a tilt (diminishing) factor γ, used to quantify the tidal tilt of the vertical, whereas the latter measured the more easily interpretable gravity change by making use of a gravimetric factor δ (Baker [27]). Both factors represent a combination of the Love numbers h and k, namely (Melchior [356])

$$\begin{array}{l} \gamma = 1 - h + k\,, \\ \delta = 1 + h - \frac{3}{2}k \end{array} \quad \Rightarrow \quad \begin{array}{l} h = 5 - 2\delta - 3\gamma\,, \\ k = 4 - 2\delta - 2\gamma \end{array} . \tag{278}$$

Measurements of γ and δ are now possible with a requested precision of better than one percent in order to deduce individual Love numbers with sufficient accuracy. Direct measurements of vertical and horizontal displacements by means of classical instrumentation remain difficult, especially if one allows for more complex (and realistic) Earth models, e.g., by introduction of the Earth's figure ellipticity and non-regular rotational behavior. In this respect, one investigates the nominal phase lag in Earth's response to the tide generating potential.

The tilt factor $\gamma_2 = 1-h_2+k_2 \approx 0.69$ can be directly related to the solution for the surface displacement $\zeta(\theta, \lambda, t)$. Originally, the derivation of the LTE was based on the dynamic relations for a fluid element of a spherical Earth (of radius $R_\oplus$) with fluid depth D, and with angular velocity of rotation of the Earth $\omega_\oplus$. The elements' horizontal velocity components u (southward) and v (eastward) are related to ζ by the continuity of fluid mass assumption, as expressed by a partial differential equation (Cartwright [93])

$$\frac{\partial}{\partial\theta}\big(vD\sin\theta\big) + \frac{\partial}{\partial\lambda}\big(uD\big) + R_\oplus \sin\theta \frac{\partial\zeta}{\partial t} = 0. \tag{279}$$

One can decompose the total tidal potential $W^{\text{total}}_{\text{tidal}}$ into a primary (tide-generating) (W_{tidal}) and a secondary (δW_{tidal}) potential, which correspond to the tidal forces as specified by Newton due to the mutual gravitational interaction of the celestial bodies and to the self-attraction of the global fluid deformation, respectively. Both parts depend on time t and the observer's position (θ, λ). The primary potential additionally depends explicitly on the luminaries' position in space, e.g. determined by its co-declination $\Theta = \pi/2 - \delta$, right ascension α, and equatorial horizontal (sine) parallax Π_0. Nominal values for Π_0 are given in various astronomical fact books (e.g. Allen [10], Cox [108]) or can be calculated based on available ephemerides.

Some remarks to clarify the relations between h, k, W_{tidal}, δW_{tidal}, and the resulting (radial) displacement:

The effect of a (tidally induced) shift of the equipotential surface through a given point that is rigidly connected to the (deformable) surface of the Earth is damped with regard to its resulting spatial displacement. This is characterized by Earth's specific (elasticity) parameter h, being the only factor in the last equation of (277). In case of perfect elasticity ($h = 1$) there would be no damping, whereas in case of perfect rigidity, i.e. inelasticity ($h = 0$), there would be no displacement. In this sense, h could be regarded as a correlation coefficient between equipotential surface shifting and (radial) displacement. An observer free-floating in space (e.g., an apart from third body gravitational attraction undisturbed Earth orbiting satellite) is affected by a changing gravitational potential, causing a direct time-varying tidal force, but it can not be used as a single test particle to directly determine h.

On the other hand, the location dependent reading of a ground-fixed gravimeter is also affected by the secondary potential (self-gravitation of the shifted masses). The displacement of masses results in an additional shift of equipotential surfaces and causes an indirect time-varying tidal force acting upon the satellite and thus changes its orbit, too. This additional gravitational attraction also depends on the Earth's material structure and can be characterized by another specific parameter k.

Both parameters h (quantifying the resistance of a certain Earth matter particle ensemble against redistribution) and k (quantifying the equivalence between the redistribution of this very ensemble with gravitational attraction) depend on location and, occasionally, will have to be derived from high-resolution Earth (density) models based on geophysical observations.

Globally valid approximative parameter values, as given above, are accurate to very few digits only. Classical ground-based instruments for tide determination (e.g., gravimeter, tiltmeter) can sense only a combination of h and k, cf. equations (278). Additional off-ground platforms (e.g., aboard air vessels or satellites) are useful to separate those parameters. Atomic clocks provide just another type of instrument for this purpose.

38 Details on the tidal potential and resulting displacements

Coming back to the LTE, balancing the rates of change of the two components of horizontal momentum relative to Earth with the applied force per unit mass yields two more dynamic relations:

$$\frac{\partial u}{\partial t} - 2\omega_\oplus \cos\theta\, v = -\frac{g}{R_\oplus}\frac{\partial}{\partial \theta}\left(\zeta - \frac{W_{\text{tidal}}^{\text{total}}}{g}\right), \qquad \frac{\partial v}{\partial t} + 2\omega_\oplus \cos\theta\, u = -\frac{1}{\sin\theta}\frac{g}{R_\oplus}\frac{\partial}{\partial \lambda}\left(\zeta - \frac{W_{\text{tidal}}^{\text{total}}}{g}\right). \tag{280}$$

The combined equations (279) and (280) constitute the basic LTE. In order to model energy dissipation one may (in a simplified approach) add (linear) friction terms $\varepsilon\partial u/\partial t$ and $\varepsilon\partial v/\partial t$, depending on a friction parameter ε, as well as some (non-linear) dynamic terms in case of shallow coastal seas to the left hand sides of equations (280).

The coefficient $2\omega_\oplus \cos\theta$ corresponds to the Coriolis frequency that accounts for the significant deflective force caused by the rotation of the earthbound coordinate system. If one neglects ocean loading and the shifting masses' self-attraction the second term in the parentheses on the right hand side of (280) had to be replaced by ζ_{equil} which corresponds to the surface displacement due to an equilibrium tide. In case of a simplified model without rotation ($\omega_\oplus = 0$) and constant depth of a single world ocean ($D = \text{const.}$), the LTE reduce to

$$\frac{\partial^2 \zeta}{\partial t^2} = gD\nabla^2\zeta - D\nabla^2 W_{\text{tidal}}^{\text{total}}. \tag{281}$$

On the other hand, (ocean) loading effects can be modeled by loading Love numbers γ'_n associated with n-th degree spherical harmonics ζ_n of ζ itself; the parentheses would read (Cartwright [93])

$$\left(\zeta - \gamma_2\frac{W_{\text{tidal}}}{g} - \sum_n \frac{3}{2n+1}\frac{\overline{\rho}_{\text{ocean}}}{\overline{\rho}_{\text{solid}}}\gamma'_n\zeta_n\right), \tag{282}$$

where the ratio of the mean densities of the oceans and the solid Earth is approximately given by (e.g. Gill [208]) $3\overline{\rho}_{\text{ocean}}/\overline{\rho}_{\text{solid}} \approx 3\cdot 1.035\, g/cm^3/\, 5.515\, g/cm^3 = 0.563$. As a remark, here the stability of the solution in terms of a harmonic development (series expansion) is guaranteed by the condition $\overline{\rho}_{\text{ocean}}/\overline{\rho}_{\text{solid}} < 1$. Harrison [233] provides details on various issues that are related to loading effects.

The implied integral of ζ itself over the whole globe is a challenging task in practice. Several methods have been employed for its computation. Today, global coverage of data is provided by satellite altimetry which is the basis of the latest generation of tidal models.

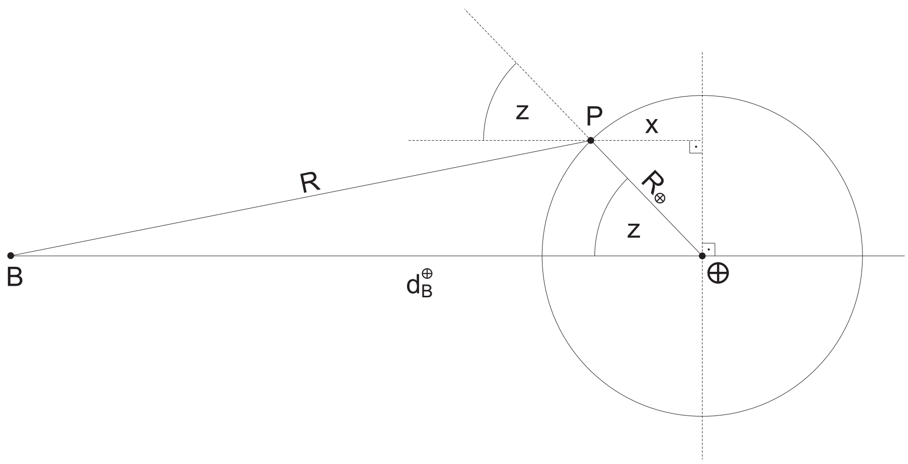

Fig. 4: Development of the tide-generating potential at a surface point P.

In compliance with the former simplifying point mass assumption for any tide-generating body B, e.g. (206) and (210), one can infer from geometrical reasoning (cf. figure 4) a first expression for the tide-generating potential at Earth's surface in Newtonian approximation as (Cartwright [93])

$$W_{\text{tidal}}^B = \Delta V_B = \frac{\mu_B}{R} - \left(\frac{\mu_B}{d_B^\oplus} + \frac{\mu_B}{d_B^\oplus}\frac{x}{d_B^\oplus}\right), \tag{283}$$

where $V_B = \mu_B/R$ with $\mu_B = GM_B$.

Next, one usually replaces x by $R_\oplus \cos z$ and $1/R = 1/\sqrt{R_\oplus^2 + d_B^{\oplus 2} - 2R_\oplus d_B^\oplus \cos z}$ by its binomial series expansion, which results in a superposition of different order terms. Retaining only the leading term yields

$$W^B_{\text{tidal}}(R_\oplus, z) = \frac{\mu_B R_\oplus^2}{2 d_B^{\oplus 3}} \big(3\cos^2 z - 1\big) \tag{284}$$

with z representing the dependency on relative position between the primary body's surface point and the location of the tide-generating body. Expressing the distance via the (sine) parallax angle $\Pi_0^B := R_\oplus / d_B^\oplus$, we find for our first order approximation

$$W^B_{\text{tidal}}(R_\oplus, z) = \underbrace{\frac{3\mu_B}{2R_\oplus}\, \Pi_0^{B\,3}}_{=:\,k_B} \big(\cos^2 z - \tfrac{1}{3}\big)\,. \tag{285}$$

Higher order terms can also produce significant, i.e. detectable, tides. Especially in case of the Moon one can not neglect those additional terms.

The corresponding force vector at P can be derived by simply taking the partial derivatives of $W^B_{\text{tidal}}(R_\oplus, z)$ with respect to $R_\oplus$ (vertical direction) and z (horizontal direction). The resulting tidal variation of gravity and the deflection of the vertical would be observed (on a perfectly rigid Earth) by a gravimeter and a tiltmeter, respectively. The tidal displacement (positive upwards) of the equipotential surface at Earth's surface relative to the center of the Earth, i.e. $W^B_{\text{tidal}}(R_\oplus, z)/g$, is called the equilibrium tide. Regarding the major two tide-generating bodies (Moon, Sun), we get the following approximative maximum peak to peak ranges (Baker [27]):

	equil. tide	tidal gravity	tidal tilt
Moon:	$53.5\,cm$	$165\,\mu gal$	$168\,nrad$
Sun:	$24.6\,cm$	$76\,\mu gal$	$77\,nrad$

(286)

Clearly, the tidal distortion due to the Moon is more than twice as much as the Sun's influence. The values for the equilibrium tide are directly related to the so-called Doodson constant $\overline{D}_B := 3\mu_B R_\oplus^2 / 4\overline{d}_B^{\oplus 3} = \overline{k}_B/2$, such that $\overline{D}_{\text{Moon}}/g = 26.75\,cm$ and $\overline{D}_{\text{Sun}} = 0.4605\overline{D}_{\text{Moon}}$ (Hendershott [245]). The displacement of an equipotential surface can refer to various points. For instance, relative to the center of the Earth it is given by $(1+k_2)W_{\text{tidal}}/g$ whereas relative to the deformed surface one has to apply the formula $(1+k_2-h_2)W_{\text{tidal}}/g$ (Baker [27]). Again, in both cases we assume a spherical non-rotating but elastic Earth.

The use of the zenith angle z is not suitable in practice. Instead, geographical coordinates (θ, λ) of P, explicit astronomical quantities (Θ_B, α_B), as well as $\omega_\oplus t$ for the relation between body-fixed and space-fixed directions shall be applied. From spherical trigonometry one gets (Doodson [143])

$$\cos z = \cos\theta \cos\Theta_B + \sin\theta \sin\Theta_B \cos\big(\alpha_B - \lambda - \omega_\oplus t\big), \tag{287}$$

where $H := \omega_\oplus t + \lambda - \alpha_B$ is referred to as the hour angle of the observer with respect to the tide-generating body (Hendershott [245]). Introducing (287) into (285) and algebraically expanding the parenthesis leads to

$$W^B_{\text{tidal}}(R_\oplus, z) = k_B\big(f_1 + f_2 + f_3\big) \tag{288}$$

with

$$\begin{aligned}
f_1 &= \tilde{f}_1 \cos\big(0(\alpha_B - \lambda - \omega_\oplus t)\big), \quad & \tilde{f}_1 &= \cos^2\!\theta \cos^2\!\Theta_B + \tfrac{1}{2}\sin^2\!\theta \sin^2\!\Theta_B - \tfrac{1}{3} = \tfrac{1}{24}\big(1+3\cos 2\theta\big)\big(1+3\cos 2\Theta_B\big), \\
f_2 &= \tilde{f}_2 \cos\big(1(\alpha_B - \lambda - \omega_\oplus t)\big), \quad & \tilde{f}_2 &= \tfrac{1}{2}\sin 2\theta \sin 2\Theta_B\,, \\
f_3 &= \tilde{f}_3 \cos\big(2(\alpha_B - \lambda - \omega_\oplus t)\big), \quad & \tilde{f}_3 &= \tfrac{1}{2}\sin^2\!\theta \sin^2\!\Theta_B\,,
\end{aligned} \tag{289}$$

such that

$$W^B_{\text{tidal}}(R_\oplus, z) = k_B\Big(\tfrac{1}{24}\big(1+3\cos 2\theta\big)\big(1+3\cos 2\Theta_B\big) + \tfrac{1}{2}\sin 2\theta\, \sin 2\Theta_B \cos H + \tfrac{1}{2}\sin^2\!\theta \sin^2\!\Theta_B \cos 2H\Big). \tag{290}$$

The first term is independent of $\omega_\oplus t$ and thus varies only with the luminaries' orbital periods, e.g., 1 month (Moon), 1 year (Sun), and so on. The second term has periods of one Earth rotation (diurnal periodicity), whereas the last term has periods of half an Earth rotation (semidiurnal periodicity), of course both with orbital modulations. Superimposed are additional modulations due to variations in parallax as contained in the common factor k_B. The mathematical reason is that longitudinal angles are measured with respect to the equator and the tidal ellipsoid (approximation) is tilted within this equatorial coordinate system. Both, angular quantities as well as distances to the tide-generating bodies show variations, such that the tidal potential will finally have asymmetrical components (Hendershott [245]).

Variations of co-declination and orbital distance in time change the potential quite systematically. The variation in co-declination is related to the precession of the equinoxes for the Sun (due to the tilt of Earth's rotational axis with respect to the ecliptic) and to the precession of the lunar (ascending) node for the Moon (due to the inclination of the Moon's orbit around Earth with respect to the ecliptic). Variations in orbital distance are due to the orbits' ellipticities.

The functions $\tilde{f}_i$ can be recognized as associated Legendre functions of degree 2. We only explicitly mentioned the primary tides that result in Laplace's classical three species f_1 (long-period), f_2 (diurnal), and f_3 (semidiurnal). Further expansion into harmonic terms would require more elaborate expressions for orbital motions in respective planes. The Moon generates significant secondary tides with an additional species $f_4 \sim \cos 3(\alpha_B - \lambda - \omega_\oplus t)$ (terdiurnal periodicity), for example.

Traditionally, variations in W^B_{tidal} are Fourier-decomposed and result in modulations of basic tidal frequencies, i.e., each species is made up of a product of different time-varying functions (Baker [27]). Generalizing (288) to (290) (separation into species), the tidal harmonics approach can be written as, cf. (276),

$$W^B_{\text{tidal}} = \sum_s W^B_{\text{tidal}_s} \quad \text{with} \quad W^B_{\text{tidal}_s} = \overline{D}_B G_s \sum_k a_k \cos\left(\sigma_k t + s\lambda + \theta_k\right), \tag{291}$$

where a_k, σ_k, and θ_k are the individual constituents' amplitude, harmonic frequency, and phase angle. The factors G_s, consistent with (289), are $G_0 = (1 - \sin^2\theta)/2$, $G_1 = \sin 2\theta$, $G_2 = \cos^2\theta$, and so on. The species are represented by $s = 0, 1, 2, \ldots$ and the harmonic frequencies are most often restricted to a linear combination of five fundamental astronomical frequencies (besides Earth rotation) which lead to the largest effects (Hendershott [245]):

$$\sigma_k = s\omega + \sum_{l=1}^{5} m^k_l f_l \tag{292}$$

with $m^k_l = 0, \pm 1, \pm 2, \ldots$. Classically, the astronomical frequencies f_l are symbolized by (Baker [27], Agnew [2]) s (mean longitude of the Moon; variation of lunar declination with period $2\pi/f_1 = 27.321582$ days, or one tropical month), h (mean longitude of the Sun; variation of solar declination with period $2\pi/f_2 = 365.242199$ days, or one tropical year), p (mean longitude of lunar perigee; variation of lunar perigee) with period $2\pi/f_3 = 8.847$ years), N' (mean longitude of lunar ascending node; variation with period $2\pi/f_4 = 18.613$ years), and p_S (mean longitude of perihelion; variation with period $2\pi/f_5 = 20941$ years). As mentioned before, one can represent the frequencies in a more compact notation by use of the Doodson numbers, where in accordance with (276) we have (Hendershott [245])

$$D_k := s\, m^k_1\, m^k_2\, m^k_3\, m^k_4\, m^k_5 + 055555. \tag{293}$$

Depending on the actual time argument that should be used as independent variable, i.e. the reference tide generating body (Moon or Sun), for ω one takes either $\omega_\oplus - f_1$ (in case of Moon, astronomical time variable τ with one mean lunar day as reference period, i.e. 24 hours 50 minutes 28.3 seconds corresponding to 14.4920521 degrees per mean solar hour) or $\omega_\oplus - f_2$ (in case of Sun, astronomical time variable t with one mean solar day as reference period, i.e., 24 hours corresponding to 15 degrees per mean solar hour).

Finally, one can allocate specific nominal values for the amplitudes and frequencies (or periods) to each tidal constituent; for the main tidal harmonics see, for instance, table 2 (equilibrium tide and rigid Earth gravity) in Baker [27] or tables 2 and 4 in Agnew [2].

As an example, the most prominent tidal harmonic, namely the semidiurnal tide M_2 (classical symbol as being in use since the times of Darwin [121] and Thomson/Tait [546]) with a period of 12.42 solar hours (due to the $\cos 2\tau$ term), also known as the semidiurnal principal lunar, produces an equilibrium amplitude of $24.3\,\text{cm}\sin^2\theta$. This value can be directly derived from equations (285) and (290) with

$$k_{\text{Moon}} \cdot \tfrac{1}{2}\sin^2\Theta_{\text{Moon}}\Big|_{\max} \sin^2\theta = \frac{3\mu_{\text{Moon}} R^2_\oplus}{4 {d^\oplus_{\text{Moon}}}^3} \cos^2\delta^{\min}_{\text{Moon}} \sin^2\theta\,, \tag{294}$$

where the maximal deviation occurs at minimal declination of the Moon, i.e., for $\delta^{\min}_{\text{Moon}} = 23.44° - 5.13° = 18.31°$ as calculated from the mean obliquity of the ecliptic and the maximum inclination of the lunar orbital plane with respect to this mean. The other nominal values used for this calculation are $\mu_{\text{Moon}} = 4902.7779\,\text{km}^3/\text{s}^2$, $R_\oplus = 6378\,\text{km}$, $d^\oplus_{\text{Moon}} = 384000\,\text{km}$. Accounting for elasticity in accordance with (277), the body-tide radial displacement amplitude solely due to M_2 is

$$a^{(M_2)}_R(\theta) = 24.3\,\text{cm} h_2 \sin^2\theta \approx 14.7\,\text{cm}\sin^2\theta. \tag{295}$$

Higher accuracy simulations (e.g., if errors should be less than 1 percent) must take into account additional effects like Earth's ellipticity, variable rotation, anelasticity, and lateral heterogeneity in internal (density) structure or even the lag of the body-tide bulge resulting in an observable gravity body-tide phase lag. The latter effect is superimposed by uncertainties in tidal loading corrections and instrumental phase lags (Baker [27], Agnew [2]).

39 Sensitivity of clocks to tidally induced potential differences

To start with simple calculations only (co-)latitude has to be fixed first. For simulations, various geographical places of potential atomic clock sites or quantum optical laboratories were selected, see table 1 (remark: many more laboratories around the world, e.g. NIST/Boulder in Colorado/USA, could be listed). Besides specialized laboratories operational fundamental Earth observation stations, e.g. at Wettzell/Bavaria, run atomic clocks, hydrogen masers and frequency combs that will take part in long range time and frequency comparison campaigns as in the proposed ACES project for instance. Instead of the very points in space where the actual optical transition within each apparatus (i.e., the atomic interrogation by use of a laser beam) takes place, the coordinates of nearby reference markers were used, e.g., in case of PTB various masonry bolts at the respective buildings that house the individual clocks. The non-trivial local tie problem is a separate one, that will have to be addressed later on by dedicated high precise in-situ surveying campaigns if real clock comparisons are to be performed. For the moment it is sufficient to calculate/simulate any effect with respect to those well-defined physical reference markers in the clocks' immediate neighborhood.

Tab. 1: Selection of laboratories that run (optical) atomic clocks and/or other frequency standards. Horizontal positions (geographical latitude and longitude) are at least accurate and precise on the (few) m-level, whereas given nominal values for vertical positions (represented here by ellipsoidal heights with respect to ITRF-08) are accurate on the cm-level.

institution	reference marker (building or lab)	frequency standard	latitude $^\circ N$	longitude $^\circ E$	height m	no.
PTB	KB01 (Kopfermann)	Yt single ion	52.29588	10.45941	119.69	P_1
	GB01 (Giebe, cellar)	In single ion	52.29577	10.46069	116.89	P_2
	LB01 (von Laue)	Al single ion	52.29725	10.46059	118.37	P_3
	PB01 (Paschen)	Sr lattice	52.29623	10.46164	120.89	P_4
SYRTE		Sr lattice	48.83635	2.33655		P_5
		Hg lattice	48.83635	2.33655		P_6
IQO/LUH		Mg lattice	52.38256	9.71878		P_7
MPQ	lab floor (D0.41)	H maser	48.25978	11.66656	522.60	P_8

The given ellipsoidal heights can be converted into physical heights, e.g., by applying reductions/corrections via corresponding (quasi-)geoid height and/or geoid undulation calculations, and additional ITRS-ETRS transformations if necessary. The latter might be done pointwise online via an EUREF permanent network web interface (EUREF [173]). As a reference height surface one may choose a local geoid model, e.g., in Germany the German Combined Geoid 2011 (GCG2011) in order to derive consistent normal heights with respect to the national reference height system DHHN92 (comprising normal gravity computation with respect to GRS80 parameters and point coordinates with respect to ETRS89). For height conversions (physical from ellipsoidal) there is also an online tool readily available (BKG [48]). In case of international clock comparison campaigns globally oriented software packages and models for the computation of geoid undulation are to be preferred, e.g., as provided by Pavlis [405] based on the Earth Gravitational Model EGM2008 (Pavlis et al. [404]).

Coming back to equation (295) exemplarily for point P_1, the individual body tide M_2 will give raise to an radial surface displacement with amplitude $a_R^{(M_2)}(\theta_{P_1} = 37.70412^\circ) \approx 5.5\,cm$. Likewise, the horizontal displacement components will be on the few cm-level even though smaller. The same holds true for other main tidal harmonics and its species, e.g., the diurnal principal lunar (symbol O_1) or lunar fortnightly (M_f) body tides.

Remark: in comparison to the southernmost latitude in table 1 the amplitude $a_R^{(M_2)}(\theta_{P_8} = 41.74022^\circ) \approx 6.5\,cm$ is different by about one centimeter, which is the equivalent difference in height for the aspired precision level $O(10^{-18})$ of modern optical clocks. Again, these crude estimates involved many simplifying assumptions, e.g., both sites placed on the surface of a sphere of radius $R_\oplus = 6378\,km$ thus neglecting any significant topographic height difference altogether. In order to perform more realistic simulations, one can apply approved software packages for the calculation of tides, e.g., the program ETGTAB (version 900527, written in FORTRAN 77, original coding done by H.G. Wenzel, some modifications added in 2010 by L. Timmen). This software comprises three optional tidal potential developments, namely the ones by Doodson [143] with 378 superimposed

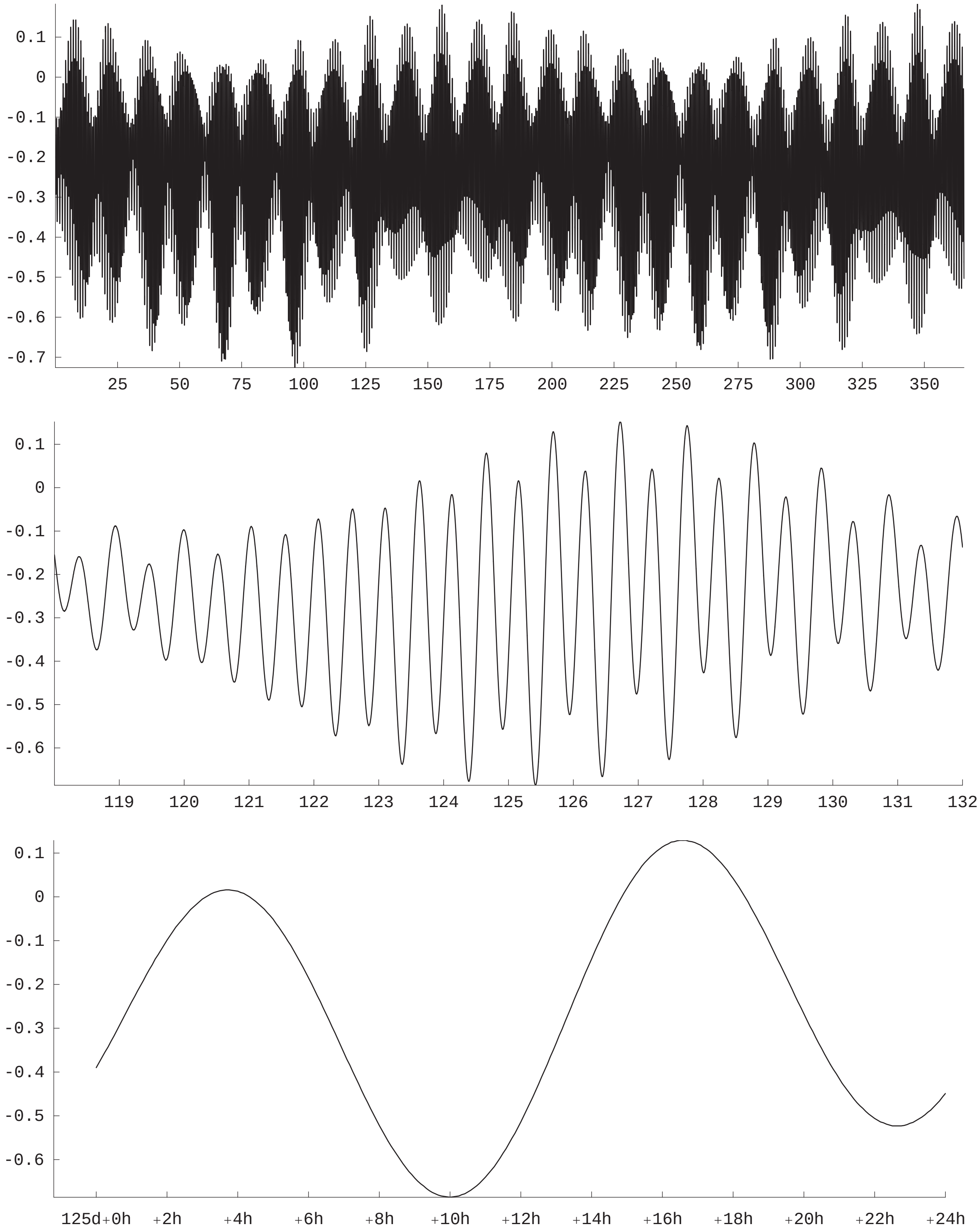

Fig. 5: Differences in tidal potential variations between points P_1 and P_8 (see table 1) in m^2/s^2 vs day of year 2012. Throughout a year the maximum peak-to-peak difference is about $0.8\,m^2/s^2$ (top). The main tidal species (semidiurnal, diurnal, long-term, e.g. fortnightly) are clearly visible (middle). In the course of a single day the maximum tidal potential difference can build up within approximately 7 hours (bottom).

individual tidal waves, Cartwright-Edden-Tayler (Cartwright/Tayler [91], Cartwright/Edden [92]) (505 waves), and Tamura [538] (1200 waves). The last mentioned model incorporates not only lunisolar influences but also accounts for some effects due to Jupiter and Venus.

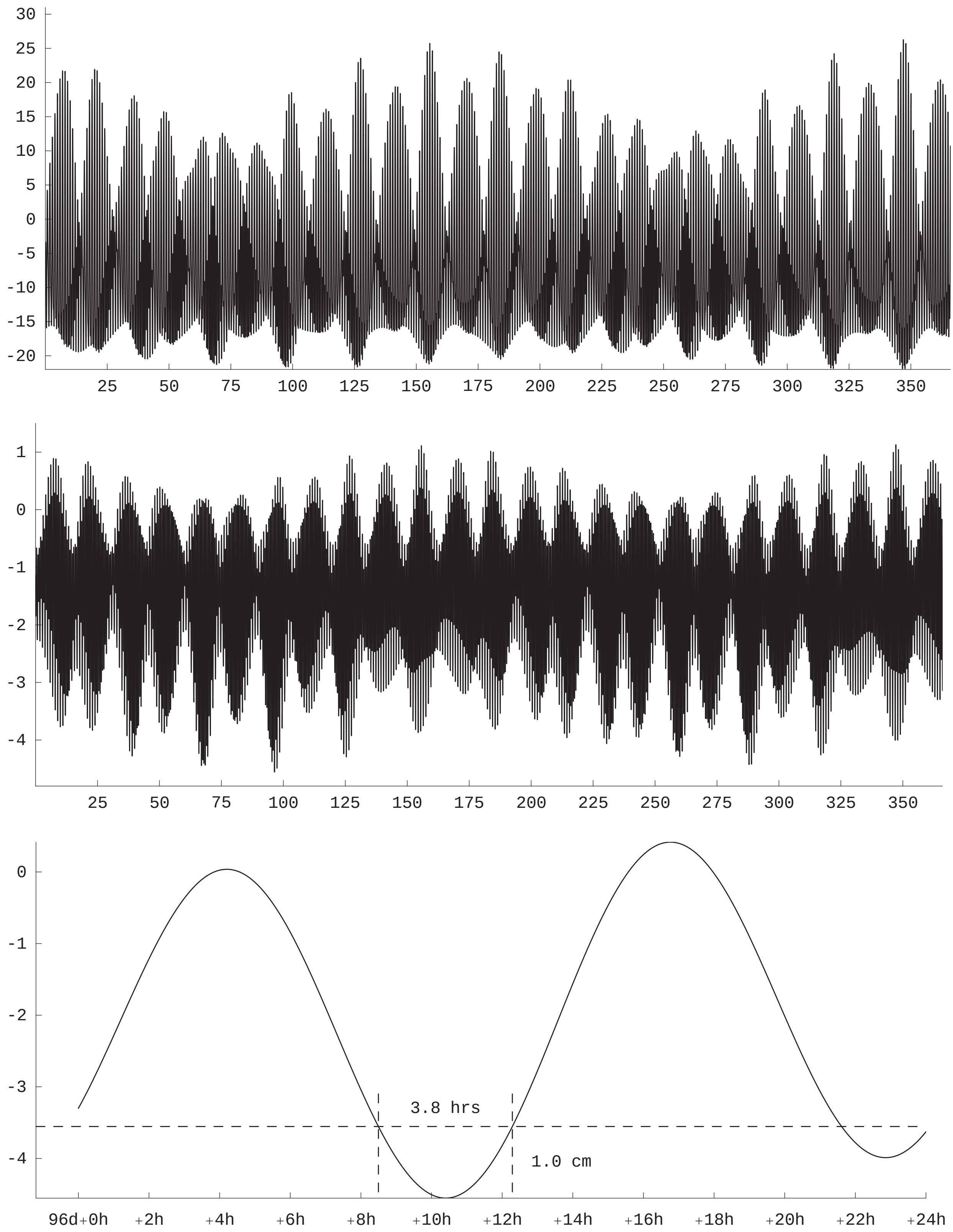

Fig. 6: Vertical displacement of points in cm vs day of year 2012. Each point, e.g. P_1, is displaced by several dm each day throughout a year (top). The differential displacement, e.g. between points P_1 and P_8, is of lower magnitude (middle). In this example, the maximum relative displacement occurs at day 96 of 2012. Within nearly 4 hours around the maximum this difference changes by a single cm (bottom).

ETGTAB can be used to compute the tidal potential itself, or its vertical component (gravity variation) and horizontal component (tidal tilt), respectively. There exist updated versions of this program, e.g., version 930821 which is available from the University of Jena (Thuringia/Germany). It comprises an additional tidal potential development model by Büllesfeld [76] (665 waves), and one can also apply an elaborate body tide computation based on the parameters (Love numbers and Shida numbers, gravimetric and tilt factors) of a specific inelastic Earth model following Wahr [567] (derivation of a globally valid averaged model from a number of different structural models), Dehant [126] (improving the elastic model by an inelastic Earth's mantle, and introduction of latitude-dependent Love numbers) and Zschau/Wang [613] (long-term implications of imperfect elasticity).

Similar software packages were developed for the special problem of detiding, i.e., the removal of tidal and atmospheric effects in deformation analysis applications (Amoruso et al. [14]). For more details and specific nominal values related to body tides and loading effects see for instance Landolt-Börnstein [318]. Seidelmann [496] in his table 4.351.1 lists amplitudes and phases of displacements of many observation sites due to ocean loading caused by the main tides.

In ETGTAB the astronomical elements are computed either by simplified formulas (approximative series expansions) of Newcomb (Sun) and Brown (Moon) or updated expressions as provided by Tamura [538]. These formulae in parts are given in various textbooks (e.g. Schödlbauer [485], Vallado/McClain [561], Seidelmann [496]).

Figures 5 and 6 illustrate the differential effect of tidal action, exemplarily shown here for two selected sites in Germany (PTB/Braunschweig and MPQ/Garching). The tidal potential development of Tamura was used in combination with the Wahr-Dehant-Zschau elastic Earth model. Clearly, the relative (vertical) displacement effect is on the *cm*-level and should thus be detectable by optical atomic clock readings. Even larger effects are expected for a comparison of sites that show a larger separation in east-west direction. As soon as still missing data on actual clocks' (vertical) positions become available, analogous calculations will be performed. From our first simulations already one can conclude that the duration of relative displacements between remote sites (at certain time intervals of the year) within the range of a single centimeter corresponds to an available averaging time that is sufficient to reach the required clock performance levels.

40 Sensitivity of clocks to the tidally induced Doppler effect

Besides the potential difference, the movement of the sites itself, i.e., its relative velocity in comparison to each other, alters the clock rates. Referring to the general Doppler effect (55), we obtain for our example (comparison PTB-MPQ at day 96 of the year 2012) a nominal relative frequency change of $\Delta f/f = 7.3 \cdot 10^{-15}$ because

$$\frac{\Delta f}{f} := \frac{|f_R - f_E|}{f_E} = \left| \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1+\frac{v_r}{c}} - 1 \right| , \tag{296}$$

where we can neglect horizontal displacements, i.e., assume $v \approx v_r$, and take $v_r = 2.18022 \cdot 10^{-6}$ m/s resulting from an absolute relative displacement change of 4.9709 cm within 22800 s $\approx$ 6.3 hrs, cf. bottom of figure 6. One could also apply equation (141) with $v \approx v_r$ to get an approximative expression for the Doppler effect as

$$\frac{\Delta f}{f} = \frac{v_r}{c} + \frac{1}{2}\left(\frac{v_r}{c}\right)^2 + O\left(c^{-3}\right) \approx 7.3 \cdot 10^{-15}. \tag{297}$$

The order of magnitude of this state-of-motion effect implies that it must not be neglected even for today's atomic clocks. Figure 7 quantifies the dependency of the general Doppler effect on v_r.

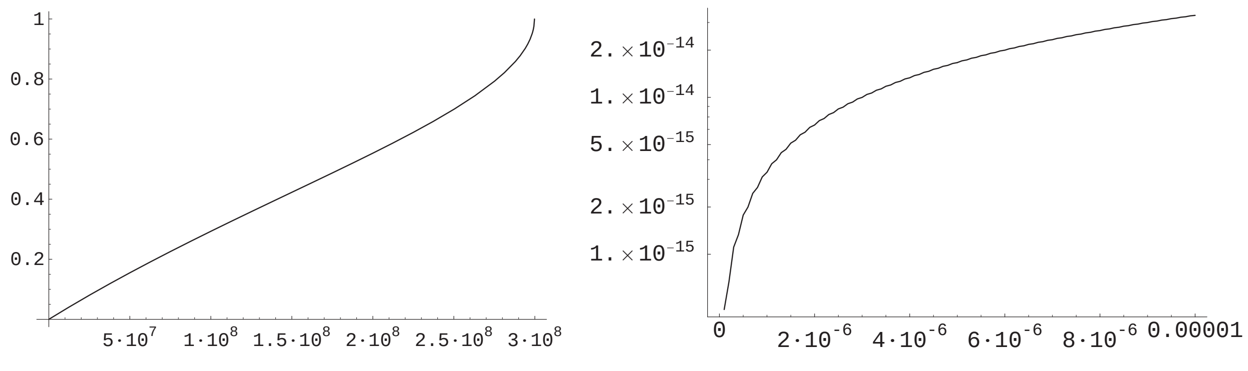

Fig. 7: Frequency shift $\Delta f/f$ due to the Doppler effect vs radial velocity v_r in m/s for $v_r \in [0, c]$ in general (left), and for expectable relative velocities due to tidally induced vertical displacements (right), respectively.

The reading of a single clock is already affected by its tidally induced displacement alone. But again, there exist many more (geophysical) effects that influence the position and velocity vector of an observation site. Besides solid Earth tides, ocean and atmospheric loading, the pole tide as well as plate and polar motion, etc. have to be taken into account. Moyer [372] provides explicit expressions for the individual displacements that may serve as a sound basis for more detailed simulations.

41 Concluding remarks on the comparison of clocks

According to equation (133) one can compare the proper times of individual clocks

$$\frac{\mathrm{d}\tau_k}{\mathrm{d}t} = \sqrt{1 - \frac{2V_k}{c^2} - \frac{v_k^2}{c^2}} = 1 - \frac{V_k}{c^2} - \frac{v_k^2}{2c^2} + O\big(c^{-4}\big), \tag{298}$$

such that for two clocks (assumed to be identical in construction) at locations with gravitational potential $V_k(\mathbf{r}_k, t)$ and velocities v_k (relative to a common reference frame) one gets

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_2} \approx \left(1 - \frac{V_1}{c^2} - \frac{v_1^2}{2c^2}\right)\left(1 + \frac{V_2}{c^2} + \frac{v_2^2}{2c^2}\right) = 1 + \frac{V_2 - V_1}{c^2} + \frac{v_2^2 - v_1^2}{2c^2} + O\big(c^{-4}\big). \tag{299}$$

Here we apply the geodetic sign convention for the (gravitational) potential, i.e., its values are always positive. At infinity the potential is zero. Equation (298) indicates that only a clock at rest with an infinite distance to all gravitational sources would show coordinate time, and run with $\mathrm{d}t$, respectively. In real world, any clock will run slower with its own/proper time step $\mathrm{d}\tau_k < \mathrm{d}t$. The time step relates to the period of the clock-specific periodical process, i.e., the occurrence of two successive wavecrests (or wave troughs), which defines a single clock tick $\mathrm{d}N$. Thus the (proper) frequencies are given by

$$\nu_k = \frac{\mathrm{d}N}{\mathrm{d}\tau_k}\,. \tag{300}$$

Consequently, any decrease of $\mathrm{d}\tau$ (clock runs slower) leads to an increase in ν, i.e., it takes more ticks to cover the same (standard/coordinate) time step $\mathrm{d}t$. Remark: For $v_k = 0$ one can also infer directly from equation (298) that the extremum $\mathrm{d}\tau_k = 0$ is valid for $V_k = c^2/2$, i.e., in case of a point mass model with $V_k = GM/r_k$, the event horizon (Lambourne [306]) (or Schwarzschild radius) is given by $r_k = 2GM/c^2 = r_S$, cf. § 17. For the Earth with $GM_\oplus = \mu_\oplus \approx 398600.442\,km^3/s^2$ we find $r_S^\oplus = 8.87\ mm$.

In case of $v_k = 0$ equation (299) reduces to

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_2} - 1 = \frac{\nu_2 - \nu_1}{\nu_1} = \frac{V_2 - V_1}{c^2}\,. \tag{301}$$

Schneider [481] discusses several other special cases of equation (298) (e.g., clocks on a rotating Earth in orbit around the Sun), the use of different reference systems, as well as its implications for clock synchronization.

Figure 8 illustrates the situation of several clocks subject to the gravitational influence of the Earth. Various simplifications are applied. For example, we assume that the clocks are either rigidly attached to the surface of the Earth (P_1, P_2) or aboard artificial Earth satellites (S_1, S_2), the orbits of which are supposed to be circular. In case of earthbound clocks one has to apply gravity potential values W_k instead of the gravitational potential, cf. § 29.14. Furthermore, any third-body perturbations (tides etc.) or non-gravitational forces are neglected. This drawing shall only support basic arguments on the correct sign of frequency comparisons, based on a few rough numbers. Fließbach [190] provides a detailed explanation of (general) relativistic effects on clock rates.

For practical clock comparisons, one has to determine the mutual velocities with care. Earthbound clocks might move not only due to Earth's rotation, if we think of mobile clocks in transportation, or environmentally induced displacements (influence of tides, plate tectonics, etc.). In this case, one would preferably operate in an Earth-centered Earth-fixed (ECEF) reference frame instead of an Earth-centered inertial (ECI) frame. Consequently, V had to be replaced by W (geopotential) within equation (298). The W-term includes the velocity effect due to Earth's rotation already, i.e., the rotation of ECEF against ECI, and the v-term represents the remaining net effect of any motions (of the clock) relative to ECEF, e.g., when operating (atomic) clocks aboard any vehicles airplanes, trains, ships, cars, and so on (Hafele/Keating [227], [228]).

Today, in order to take the velocity-dependent relativistic effects into account, the trajectory/state of a clock can be determined accurately via GNSS or INS (inertial navigation systems based on accelerometers and gyroscopes).

Of course, GNSS systems itself (like GPS) are significantly influenced by a superposition of relativistic effects (gravitational redshift, time dilatation due to the Doppler effect, Sagnac effect due to (Earth's) rotation in

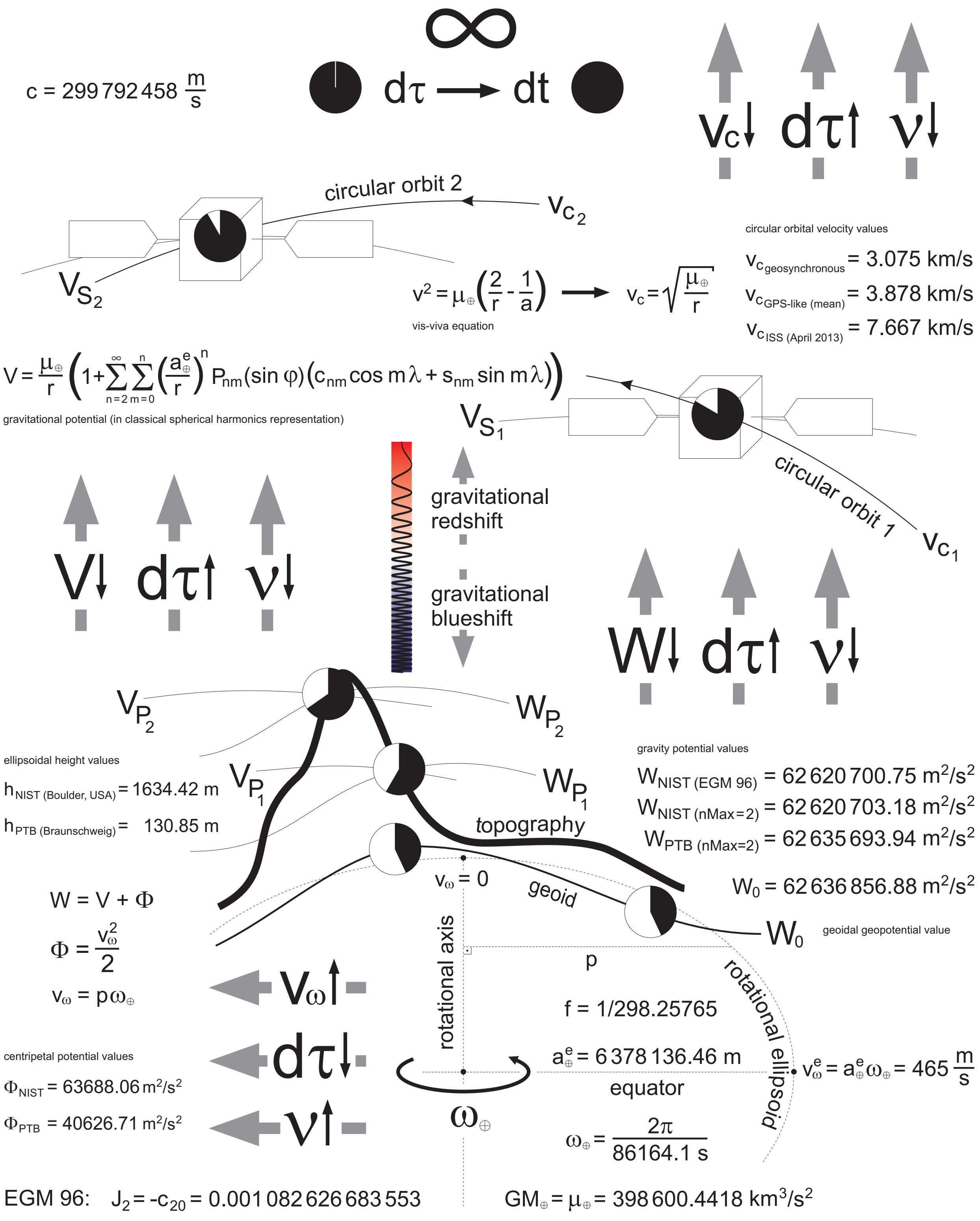

Fig. 8: Clock rates in dependence on the clocks' location and state of motion. For details see main text.

combination with a limited speed of the GNSS signal propagation) and one has to consider the net effect on its signals, i.e., a resulting frequency shift (Kleppner et al. [283], Eardley et al. [148], Ashby [21], Combrinck [106]).

Following the classical concept of height and geopotential numbers, cf. equations (267) and (268), the gravity potential $W_k = V_k + \Phi_k$ decreases with height. At the geoid ($W_k = W_0$) we define $H = 0$ such that $H_P > 0$ for $C_P = W_0 - W_P > 0$. Likewise, the centripetal potential $\Phi_k = \omega^2 r_k^2 \sin^2 \theta_k$ at the body-fixed location increases with growing distance $d = r \sin \theta$ from the rotational axis of the body, such that $\Phi_{\max} = \omega^2 a^{e2}/2$ would result at the equator of an oblate two-axial ellipsoid of revolution (which might act as a first approximation for Earth's geometrical shape). The position vector $\mathbf{r}_k$ in an earth-fixed frame is given by spherical coordinates $(r_k, \theta_k, \lambda_k)$ (radial distance r, co-latitude θ, longitude λ) via

$$\mathbf{r}_k = \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} = \begin{pmatrix} r_k \sin\theta_k \cos\lambda_k \\ r_k \sin\theta_k \sin\lambda_k \\ r_k \cos\theta_k \end{pmatrix} \quad \Leftrightarrow \quad \begin{aligned} r_k &= \sqrt{x_k^2 + y_k^2 + z_k^2}\,, \\ \theta_k &= \arccos(z_k/r_k)\,, \\ \lambda_k &= \arctan(y_k/x_k)\,. \end{aligned} \tag{302}$$

Pavlis/Weiss [403] provide ITRF94 coordinates for a reference point at the US national metrological institute NIST (National Institute of Standards and Technology, formerly known as the National Bureau of Standards), which operates (primary) frequency standards that take part in the TAI realization.

Each of the worldwide distributed TAI-defining clocks realizes its own SI second. In order to compare the clocks with each other, one had to exchange signals for synchronization/syntonization purposes, or one can apply location and state dependent clock corrections to the proper clock readings such that they relate to a common reference. For the latter option one naturally chooses the geoid, supposing that a virtual standard clock would be located on the geoid. Consequently, for any real (earthbound) clock one has to determine the frequency shift

$$\frac{f_k - f_0}{f_0} = \frac{W_k - W_0}{c^2}\,. \tag{303}$$

Remark: henceforth, we will use the symbol f for (proper) frequencies, purely to ensure compatibility with the majority of other publications. Regarding the sign, equation (303) is consistent with Bjerhammar [46], but in the denominator on the left hand side he introduces the mean $f = (f_k + f_0)/2$ for some unknown reason. In fact, the difference $\Delta f := f_k - f_0$ will be smaller than both values itself by many orders of magnitude, such that the deviation between the resulting ratios $\Delta f / f_0$ and $\Delta f / f$ is negligible. On the other hand, Pavlis/Weiss [403] apply the opposite sign, i.e., $\Delta f_{[403]} := f_0 - f_k$, interestingly enough referencing to Bjerhammar [46], too.

Obviously, the sign mismatch is due to a conflicting viewpoint regarding the frequency shift as such. Either one treats it as an error or deviation of the clock reading in comparison to an „error-free“ situation (without relativistic effects), or one wants to compute it as a correction to keep/achieve a nominal frequency. To avoid confusion, apart from a stringent relativistic reasoning chain along world lines/geodesics etc., it is probably best to clarify this issue based on a few specific/exemplary numbers, cf. figure 8.

If a clock is located above the geoid, we have $W_k < W_0$ and thus $\mathrm{d}\tau_k > \mathrm{d}\tau_0$ (clock at $\mathbf{r}_k$ runs faster than the (virtual) standard clock on the geoid, indicated by the various partially filled clock dials as sketched in the figure), or $f_k < f_0$. Again, assuming identical constructions, each clock realizes its own SI second. An observer on the geoid wants to compare his clock with the higher one. The latter sends its frequency signal towards the geoid. During its course through the gravity potential of the Earth (which increases along the path of travel in compliance with the geodetic sign convention) this signal gains some finite amount of energy, and consequently its (proper) frequency continuously increases as it travels downwards - it undergoes a (gravitational/general relativistic) blueshift. The received signal then appears to the observer on the geoid to have a higher ticking rate than his own one; his clock obviously runs slow in comparison. Conversely, a signal that is sent upwards, loses energy and its (proper) frequency decreases (gravitational redshift). As mentioned before, this frequency shift can be explained purely based on energy considerations by making use of the Mössbauer effect (Wegener [571]).

If the time signals of all (TAI-defining) clocks (above the geoid) shall be of the same frequency f_0 once they arrive at the geoid, we now know that their signals' frequencies had purposely to be set low prior to sending (to make up for the energy gain during propagation) by adding a frequency shift to the (proper) frequencies f_k according to equation (303). With indices s, r denoting sending and receiving, respectively, we take the viewpoint of a frequency correction, and thus conclude

$$f_k^s = \underbrace{\left(1 + \frac{W_k - W_0}{c^2}\right)}_{<1 \text{ for } W_k < W_0} f_0^r\,. \tag{304}$$

To present an example, we consider two frequency standards, specifically well marked/surveyed reference points in their neighborhoods, at NIST (Boulder, Colorado, USA) and PTB (Braunschweig, Niedersachsen, Germany). Both places are well above the geoid, where Boulder is of larger physical (as well as geometrical/ellipsoidal) height than Braunschweig. Following Pavlis/Weiss [403], for a certain marker at NIST, we find

$$\mathbf{r}_{\mathrm{NIST}}^{\mathrm{ITRF94}} = \begin{pmatrix} -1288394.075\,m \\ -4721673.869\,m \\ 4078630.782\,m \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} r_{\mathrm{NIST}} &= 6370980.495\,m\,, \\ \theta_{\mathrm{NIST}} &= \;\; 50^\circ 11' 38''\!.745\,, \\ \lambda_{\mathrm{NIST}} &= 254^\circ 44' 14''\!.541\,. \end{aligned} \tag{305}$$

As a remark: it is stated that these Cartesian coordinates are expected to be accurate only to 20 *cm* (or better).

Likewise, we pick a certain marker at PTB. Feldmann [180] performed GPS surveys of some laboratory rooftop markers (which are not identical to the points given in table 1, but they are located in the neighborhood), e.g.,

$$\mathbf{r}_{\mathrm{PTB}}^{\mathrm{ITRF2000}} = \begin{pmatrix} 3844056.75\,m \\ 709664.09\,m \\ 5023131.72\,m \end{pmatrix} \quad \Rightarrow \quad \mathbf{r}_{\mathrm{PTB}}^{\mathrm{ITRF94}} = \begin{pmatrix} 3844056.76\,m \\ 709664.09\,m \\ 5023131.70\,m \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} r_{\mathrm{PTB}} &= 6364923.221\,m\,, \\ \theta_{\mathrm{PTB}} &= 37^\circ 53' 24.''540\,, \\ \lambda_{\mathrm{PTB}} &= 10^\circ 27' 35.''262\,, \end{aligned} \tag{306}$$

where we applied a (7-parameter Helmert) transformation between the Earth-fixed reference frames ITRF2000 and ITRF94 (IGN [258]) to be consistent with the NIST data.

The resulting centripetal potential values for $\omega_\oplus = 7.29211505392569 \cdot 10^{-5}\,rad/s$ are

$$\Phi_{\mathrm{NIST}} = 63\,688.06\,m^2/s^2 \qquad \text{and} \qquad \Phi_{\mathrm{PTB}} = 40\,626.71\,m^2/s^2. \tag{307}$$

Thus, according to equation (299) with setting $V_k = 0$, an atomic clock at NIST would run slower than a clock at PTB, i.e. $\mathrm{d}\tau_{\mathrm{NIST}} < \mathrm{d}\tau_{\mathrm{PTB}}$, as long as we only consider the effect of Earth's rotation.

Next, we determine gravitational potential values V_k for both sites. The classical spherical harmonics representation reads (Rapp/Pavlis [443], Torge [550])

$$V_k(r_k, \theta_k, \lambda_k) = \frac{GM}{r_k}\left(1 + \sum_{n=2}^{\infty}\left(\frac{a^e}{r_k}\right)^n \sum_{m=0}^{n} P_{nm}(\cos\theta_k)\Big(c_{nm}\cos m\lambda_k + s_{nm}\sin m\lambda_k\Big)\right), \tag{308}$$

where $P_{nm}(\cos\theta) = P_{nm}(\sin\phi)$ denote the (co-)latitude-dependent associated Legendre functions of degree n and order m. Pavlis/Weiss [403] employed the complete Earth Gravity Model EGM96 ($n_{\max} = m_{\max} = 360$) for the spherical harmonic coefficients c_{nm} and s_{nm}, which are given in a normalized form

$$\begin{aligned} \bar{c}_{nm} &= N_{nm} c_{nm}\,, \\ \bar{s}_{nm} &= N_{nm} s_{nm} \end{aligned} \qquad \text{with} \qquad N_{nm} = \sqrt{\frac{(n+m)!}{(2-\delta_{0m})(2n+1)(n-m)!}}\,. \tag{309}$$

Here, for our rough estimate, it is sufficient to retain only the most dominant zonal degree-2 term (which, in case of the Earth, is responsible for more than 90 percent of its mass inhomogeneities) with $\bar{c}_{20} = -0.000484165371736$ or $J_2 := -c_{20} = -\sqrt{5}\,\bar{c}_{20} = 0.00108262668355$. Using the approximative expression

$$V_\oplus(\mathbf{r}) \approx \frac{\mu_\oplus}{r}\left(1 + c_{20}\left(\frac{a^e_\oplus}{r}\right)^2\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)\right) \tag{310}$$

with $\mu_\oplus = GM_\oplus = 398600.4418 \cdot 10^9\,m^3/s^2$ and $a^e_\oplus = 6378136.46\,m$ yields

$$V_{\mathrm{NIST}} = 62\,557\,015.12\,m^2/s^2 \qquad \text{and} \qquad V_{\mathrm{PTB}} = 62\,595\,067.23\,m^2/s^2. \tag{311}$$

Following equation (301), the pure effect of a gravitational frequency shift would make a clock at NIST run faster than a clock at PTB.

In total, we get the respective gravity potential values

$$W_{\mathrm{NIST}} = 62\,620\,703.18\,m^2/s^2 \qquad \text{and} \qquad W_{\mathrm{PTB}} = 62\,635\,693.94\,m^2/s^2. \tag{312}$$

Taking the full set of spherical harmonic coefficients leads to $W_{\mathrm{NIST}} = 62\,620\,700.75\,m^2/s^2$ (Pavlis/Weiss [403]), which implies that, at least for highly precise clock comparisons, an approximation of the Earth's gravitational potential by the mass monopole and quadrupole moments is not sufficient, because (cf. equation (303))

$$\frac{\delta f}{f} = \frac{\delta W}{c^2} \qquad \Rightarrow \qquad \delta W \approx 3\,m^2/s^2 \quad \to \quad \frac{\delta f}{f} \approx 3.3 \cdot 10^{-17}. \tag{313}$$

Such an error magnitude is well in the frequency stability range of the best performing (optical) clocks. Or, from a practitioners point of view, upcoming atomic clocks are sensitive to higher order mass multipole moments and thus could provide a supplementary tool for gravity field monitoring purposes. So far, this task is mainly driven by costly space missions that have to be launched every few years due to its satellites' limited lifetimes. In contrast, a continuously running globally distributed clock network in future may provide uninterrupted and consistent time series of data on the Earth's gravity field, valuable to the proposed GGOS.

In an approximative manner, one can apply an alternative series expansion for the computation of the values W_k, based on § 32, cf. equation (267),

$$W_k = W_0 - g(\phi_k) h_k + \cdots\,, \tag{314}$$

where $g(\phi)$ is the gravity acceleration on the geoid in dependence on geographical latitude (Soffel/Langhans [514])

$$g(\phi) \approx \left(9.78027 + 0.05192 \sin^2\phi\right) m/s^2, \tag{315}$$

and h denotes height above the geoid.

The ratio $\mathrm{d}\tau_{\mathrm{NIST}}/\mathrm{d}\tau_{\mathrm{PTB}}$ can thus be written as

$$\frac{\mathrm{d}\tau_{\mathrm{NIST}}}{\mathrm{d}\tau_{\mathrm{PTB}}} = \frac{f_{\mathrm{PTB}}}{f_{\mathrm{NIST}}} = \frac{1 - \dfrac{W_0 - g(\phi_{\mathrm{NIST}})h_{\mathrm{NIST}}}{c^2}}{1 - \dfrac{W_0 - g(\phi_{\mathrm{PTB}})h_{\mathrm{PTB}}}{c^2}} \approx \frac{1 + \dfrac{g(\phi_{\mathrm{NIST}})h_{\mathrm{NIST}}}{c^2}}{1 + \dfrac{g(\phi_{\mathrm{PTB}})h_{\mathrm{PTB}}}{c^2}}\,. \tag{316}$$

For our estimates we will approximate the geoid by an ellipsoid of revolution with given size and shape values

$$a = a^e_\oplus = 6\,378\,136.46\,m \text{ (semi-major axis)} \qquad \text{and} \qquad f = 1/298.25765 \text{ (flattening)}\,, \tag{317}$$

which represents an ideal Earth ellipsoid in the tide-free system, whereas the GPS-related World Geodetic System (WGS-84) applies nominal values $a = 6378137.0\,m$ and $f = 1/298.2572235630$. Remark: GNSS provide ellipsoidal heights, e.g., GPS heights relate to WGS-84.

The transformation between Cartesian coordinates (x, y, z) and ellipsoidal geographical coordinates (ϕ, λ, h) (Heck [237]), where a and f are given parameters ($b = a(1 - f)$), can be performed via (Mittermayer [367])

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ \left(\dfrac{N}{1+e'^2} + h\right)\sin\phi \end{pmatrix} \quad \Leftrightarrow \quad \begin{aligned} \phi &= \arctan\left(\frac{z + e'^2 b \sin^3 t}{p - e^2 a \cos^3 t}\right), \\ \lambda &= \arctan(y/x)\,, \\ h &= \frac{p}{\cos\phi} - \frac{a}{\sqrt{1 - e^2\sin^2\phi}}\,, \end{aligned} \tag{318}$$

with the auxiliary quantities

$$p = \sqrt{x^2 + y^2}\,, \quad t = \arctan\left(\frac{z\,a}{p\,b}\right), \quad c = \frac{a^2}{b}\,, \quad e^2 = \frac{a^2 - b^2}{a^2}\,, \quad e'^2 = \frac{a^2 - b^2}{b^2}\,, \quad V^2 = 1 + e'^2\cos^2\phi\,, \quad N = \frac{c}{V}\,. \tag{319}$$

The transformation $(x, y, z) \to (\phi, \lambda, h)$ has no exact analytical/closed-form solution, thus the expressions for ϕ and h on the right hand side of equation (318) actually result from series expansions. If higher accuracies are required, the following iterative procedure can alternatively be applied:

$$\begin{aligned} N^{(i)} &= \frac{a}{\sqrt{1 - e^2\sin^2\phi^{(i-1)}}}\,, \\ h^{(i)} &= \frac{p}{\cos\phi^{(i-1)}} - N^{(i)}\,, \\ \phi^{(i)} &= \arctan\frac{z}{p\left(1 - \dfrac{e^2 N^{(i)}}{N^{(i)} + h^{(i)}}\right)} \end{aligned} \qquad \text{for} \quad i = 1, 2, 3, \ldots \quad \text{with} \quad \phi^{(0)} = \arctan\frac{z}{p\,(1 - e^2)}\,. \tag{320}$$

In our example, cf. equations (315) and (316), we find

$$\begin{aligned} \phi_{\mathrm{NIST}} &= 39°59'42''.861\,, & \phi_{\mathrm{PTB}} &= 52°17'46''.391\,, \\ \lambda_{\mathrm{NIST}} &= 254°44'14''.541\,, & \lambda_{\mathrm{PTB}} &= 10°27'35''.262\,, \\ h_{\mathrm{NIST}} &= 1634.421\,m\,, & h_{\mathrm{PTB}} &= 130.848\,m\,, \\ g_{\mathrm{NIST}} &= 9.80172\,m/s^2, & g_{\mathrm{PTB}} &= 9.81277\,m/s^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathrm{d}\tau_{\mathrm{NIST}} &\approx \left(1 + 1.64\cdot 10^{-13}\right)\mathrm{d}\tau_{\mathrm{PTB}}\,, \\ f_{\mathrm{PTB}} &\approx \left(1 + 1.64\cdot 10^{-13}\right) f_{\mathrm{NIST}}\,, \end{aligned} \tag{321}$$

which is consistent with the (more precise) result according to relations (303) and (312)

$$\frac{f_{\mathrm{PTB}} - f_{\mathrm{NIST}}}{f_{\mathrm{NIST}}} = \frac{W_{\mathrm{PTB}} - W_{\mathrm{NIST}}}{c^2} \qquad \Rightarrow \qquad f_{\mathrm{PTB}} \approx \left(1 + 1.67\cdot 10^{-13}\right) f_{\mathrm{NIST}}\,. \tag{322}$$

From $\mathrm{d}\tau_{\mathrm{NIST}} > \mathrm{d}\tau_{\mathrm{PTB}}$ we conclude that the clock at NIST runs faster than an equally constructed clock at PTB. After one day this effect would lead to an accumulated time display difference of about 14.4 nanoseconds, or about 5.26 microseconds after one year, respectively. In this example, the frequency shift due to a gravitational potential difference (ΔV) is predominant in comparison to the frequency shift due to a centripetal potential difference ($\Delta\Phi$). The same holds true for another example, where one would compare the clock at PTB with another one on top of Mt. Everest. In this situation, the higher clock would run faster too (even more so) and, again accumulated during the time span of one year, its time display offset would be nearly twice as large in comparison to the NIST case (Soffel/Langhans [514]).

Regarding the task of TAI realization, all of the above-mentioned (imaginary) clocks run faster than a standard clock on the geoid, because $W_k < W_0 \Rightarrow \mathrm{d}\tau_k > \mathrm{d}\tau_0$. Frequency corrections had to be applied so that they effectively beat at the same rate as the standard clock. At NIST and PTB these corrections would amount to -15.5 and -1.1 nanoseconds per day ($1\,d = 86400\,s$), respectively, because

$$\frac{f_{\mathrm{NIST}} - f_0}{f_0} = -1.79761 \cdot 10^{-13} \qquad \text{and} \qquad \frac{f_{\mathrm{PTB}} - f_0}{f_0} = -1.29395 \cdot 10^{-14}. \tag{323}$$

What happens (to the sign of the frequency correction) if we raise the clocks' altitude ? The answer depends on the clocks' actual state of motion. In order to narrow down the vast amount of special cases, we do not consider any propelled motion here but will restrict ourselves to the free-fall motion of artificial Earth satellites. Therefore, the clock is not subject to Earth's classical centripetal potential, and equation (298) is to be used. Furthermore, we simply apply the (non-relativistic) Kepler problem - we assume an idealized two-body motion.

As an example, we consider a clock in a GPS-like orbit (Mai [337]) but with zero eccentricity (circular orbit), i.e. $r = \bar{a}_{\mathrm{GPS}} \approx 26500\,km$. From the vis-viva equation (Vallado/McClain [561]) one gets the circular velocity

$$v^2 = \mu_\oplus \left(\frac{2}{r} - \frac{1}{a}\right), \quad a = r \qquad \Rightarrow \qquad v^{\mathrm{circular}}_{\mathrm{GPS\text{-}like}} =: v_{\mathrm{GPS}} = \sqrt{\frac{\mu_\oplus}{\bar{a}_{\mathrm{GPS}}}} = 3878\,m/s\,. \tag{324}$$

Due to the comparatively high altitude of the orbit, the series expansion for the Earth's gravitational potential can be truncated after a few terms. Remark: in the early days, the GPS on-board software routines made use of an 8×8-gravity field model ($n_{\max} = m_{\max} = 8$) for in-situ orbital determination purposes. Here, we will even retain only the mass monopole term, such that

$$V(\mathbf{r}_{\mathrm{GPS\text{-}like}}) =: V_{\mathrm{GPS}} \approx \frac{\mu_\oplus}{\bar{a}_{\mathrm{GPS}}} = 15\,041\,526.11\,m^2/s^2. \tag{325}$$

Considering the situation from the point of view of the ECI frame, the comparison of our space-based clock with a stationary clock at one of the poles on the rotating geoid (where $\Phi_0 = 0 \Rightarrow V_0 = W_0$) yields, cf. equation (299),

$$\frac{\mathrm{d}\tau_{\mathrm{GPS}}}{\mathrm{d}\tau_0} = \frac{f_0}{f_{\mathrm{GPS}}} = 1 + \frac{W_0 - V_{\mathrm{GPS}}}{c^2} - \frac{v^2_{\mathrm{GPS}}}{2c^2} = 1 + 5.296 \cdot 10^{-10} - 0.837 \cdot 10^{-10} = 1 + 4.459 \cdot 10^{-10}. \tag{326}$$

Ashby [21] performs a similar estimation of the net effect, assuming a clock on the equator instead and retaining only the J_2-term in the calculation of the geopotential at the ground site, resulting in $\Delta f/f = 4.465 \cdot 10^{-10}$. Spilker [518] mentions a more general case which also comprises the normal Doppler effect and higher order terms $O(c^{-3})$ etc. It is stated that, on average, at earthbound GPS receivers the observed fractional frequency shift $(f_r - f_t)/f_t$ between transmitted (f_t) and received (f_r) GPS signals would be $4.479 \cdot 10^{-10}$. Seemingly, this result was obtained by simple point mass considerations, because a separate statement refers to the Earth's oblateness and the Sun's potential, the combined effects of which will lead to frequency shift variations between $4.458 \cdot 10^{-10}$ and $4.502 \cdot 10^{-10}$. According to Kleppner et al. [283], the lunar influence is negligible altogether.

These statements were justified in the 1970's, but in the early 1980's Moyer [370], [371] identified various lunisolar effects (and even the influence of Jupiter and Saturn) to be significant for time transformations/comparisons, especially in space geodetic techniques like VLBI, cf. equations (142) and (143) in § 30.1. Regarding the GPS, Ashby [21] discusses additional effects due to the ellipticity of its satellite orbits (the mean eccentricity is $\bar{e}_{\mathrm{GPS}} \approx 0.005$), or due to necessary orbital adjustments.

Remark: GPS-like orbits exhibit (deep) resonance effects because of its near 2 : 1 commensurability with Earth's rotation. This causes a violation of the predictions of the classical orbital perturbation theory of Kaula [271] which, among others, states that the semi-major axis shows no secular trends. In practice, the GPS operators have to perform (costly) station-keeping maneuvers at regular intervals (nearly every 11 months). Otherwise the drift of the semi-major axes would grow too strongly and finally destroy the topology of the GPS constellation, which is crucial for the functionality of the whole navigation system. Alternative orbital theories have been developed (e.g. Cui [111]) and tested (Mai [337]) that take resonance and coupling effects into account.

The ellipticity of GPS orbits gives raise to a periodic frequency shift. If neglected, it can cause timing errors of a few nanoseconds, corresponding to positioning errors of a few meters. The receiver has to account for this effect, it is not included in the navigation message due to historical reasons (Ashby [21]). Most GPS users will employ an ECEF frame, thus the receiver software additionally has to care for the Sagnac correction (Combrinck [106]).

Following equation (326), the satellite clock runs faster than the clock on the ground. The GPS system design intended the nominal frequency on the geoid to be $f_0 = 10.23\,\mathrm{MHz}$. Consequently, the GPS signal requires a transmission frequency f_{GPS} that is set low to

$$f_{\mathrm{GPS}} = \frac{10.23\,\mathrm{MHz}}{1 + 4.459 \cdot 10^{-10}} = 10.229\,999\,995\,44\,\mathrm{MHz}\,, \tag{327}$$

which is close to the values in Spilker [518] (10.229 999 995 45 MHz) and Combrinck [106] (10.229 999 995 43 MHz).

According to Ashby [21], a corresponding „factory frequency offset“ was applied to the satellites' atomic clock frequencies for compensation before launch, but only in case of the older satellites. These days a new procedure is in use: clock frequencies are measured after orbit insertion and necessary corrections are transmitted to the receivers via the navigation message.

Remark: all of our calculations are rough estimates. For any real-world clock comparisons/corrections, besides idealized major relativistic effects and various (systematic) error sources for the clocks itself, one has to consider distortions of the signals that result in path deflections, time delays, etc. The significance of minor relativistic effects (cf. § 29.13) had to be studied in detail, depending on the proposed application. For earthbound atomic clocks, major geophysical effects like Earth tides must not be neglected, as we already demonstrated in previous sections. Obviously, more realistic simulations require an elaborate consistent modeling.

Outlook

Future geodesy will benefit from a consistent incorporation of relativistic and quantum mechanic aspects. Technological progress opens the field of relativistic geodesy, where atomic clocks and other quantum engineering sensors are sensitive enough to exploit relativistic effects even in case of only moderate velocities and potential differences as usually experienced in earthbound applications.

Successful implementation in practice requires a sound theoretical basis. This means that, depending on the actual given tasks, post-Newtonian approximations of gravity may have to be extended to required higher order terms. The underlying field equations must be solved for more general real-world physical systems, e.g., extended massive non-rigid bodies with irregular shapes, density distributions, and rotational rates. Appropriate solution strategies have to be developed based on consistent reference systems for space and time.

Regarding practical aspects, the optimal configuration of clock networks and clock reading comparisons has to be investigated, depending on the needs of proposed geodetic applications. Highly precise frequency and time transfer methods are already being tested between established sites of remote metrological institutes or laboratories of physics. Experience with advanced optical fiber networks and satellite based connections, either via microwave or laser links, can be gathered from past experiments and will grow in future. Various experiments on these issues are running, others are at least proposed or in planning phase.

Certain geodetic applications rely on mobile measurement devices. Consequently, the mobility of (optical) clocks is a prerequisite for their widespread use in geodesy. Miniaturization is another important request. Various applications imply different requirements on such parameters as short/long term stability (i.e. necessary averaging time), power consumption, weight, robustness, manageability of the whole instrumentation, costs, and so on. For instance, ultra-stable oscillators, readily available for short-period applications, are much cheaper than long-term stable hydrogen masers. In the distant future, multiple miniature clocks or even chip traps (clocks on a chip) with sufficient stability might become realizable. This would enable quite different measurement setups like clock swarms, etc.

As long as an area-wise use of atomic clocks remains fiction, its selective or point-wise application stays in focus. In this sense, highly precise frequency standards are predestined for creating references in space-time. Terminologically, we should speak of reference events in space-time instead of reference points (in space). Geodetic entities like 'coordinates' or 'heights' refer to a certain reference system or surface. Furthermore, in today's classical geodesy, near real-time availability of final results plays an increasingly important role. Relativistic geodesy, based on consistent redefinitions of reference systems and surfaces, e.g. the relativistic geoid, would offer a clear and transparent way of meeting these requirements.

For the time being, the most obvious application is a clock based determination of large-scale height differences in combination with complementary geodetic measurements. The latter are necessary to resolve the local tie problem in case of co-location sites that host different geodetic techniques. Combination with data from other sources, i.e., ground-based gravimetry or space-based gradiometry, shall be used to ensure comparability with gravity field information and height system definitions. It enables the control of existing gravimetric geoids and leveling results. Chronometric leveling will provide an effective and unique way to link isolated regional geoids or tide gauges, e.g., in case of remote islands, which is a prerequisite for any profound judgement on regional consequences of long-term global sea level changes.

The chronometric approach constitutes an independent measurement technique. Continuously running globally distributed atomic clocks (clock networks) will allow for a consistent and near real-time Earth system monitoring of geophysical phenomena with high temporal resolution. Regarding spatial resolution, provided that reliable and robust transportable optical clocks with shorter interrogation times become available, they can be used to densify existing information content where other techniques may be limited in the gathering of the required data. This new technique might be more cost effective, depending on the given task, especially in comparison to high-budget space missions. On the other hand, the success of relativistic geodesy relies on further progress in quantum engineering in order to release the long-desired instrumentarium from the laboratories to the geodetic community, and on the widespread use of a consistent mathematical framework among the users.

Close cooperation between geodesists and clock operators/developers at the (metrological) laboratories will help to detect potential sources of error (methodological and/or technological) that may hamper the successful application of clocks in relativistic geodesy. Different views of perspective provide a chance to identify various new applications beyond the quite obvious chronometric leveling idea.

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