Zufallswanderer auf einem Ring

Vergleich zwischen Mastergleichung und Markov-Kette für lange Laufzeiten

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Markov-Kette

$$P(x_m, t_n + \tau) = p \cdot P(x_m - a, t_n) + q \cdot P(x_m + a, t_n)$$

$$x_m = am \qquad m = 0, 1, 2...M - 1$$

$$t_n = \tau n \qquad n = 0, 1, 2...$$

$$x_M = aM = L$$

$$P(x_m, t_n) = P(x_m + L, t_n)$$

$$P(x_m, t_0 = 0) = \delta_{x_m, x_0}$$

Explizite Lösung der Markov-kette

Fouriertransformation:

$$\tilde{P}(k, t_n) = \sum_{m=0}^{M} P(x_m, t_n) e^{ikx_m}$$

$$P(x_m, t_n) = \frac{1}{M} \sum_{k} \tilde{P}(k, t_n) e^{-ikx}$$

Die explizite Lösung lautet dann:

$$P(x_m, t_n) = \sum_{k} e^{\lambda'_k t_n} [\sin(\lambda''_k t_n) \sin(k(x_m - x_0)) + \cos(\lambda''_k t_n) \cos(k(x_m - x_0))]$$

$$\lambda'_k = \frac{1}{\tau} \ln(\sqrt{\cos k^2 + \Delta^2 \sin k}) \qquad \Delta = p - q$$

$$\lambda''_k = \frac{1}{\tau} \arctan(\Delta \tan k)$$

Bildung des Zeitlichen Mittels über zwei aufeinanderfolgende Zeitschritte

$$
\begin{aligned}
\overline{P}(x_m, t_n) &= \frac{1}{2}(P(x_m, t_n) + P(x_m, t_n + \tau) \\
&= \frac{1}{2M}\Big(\sum_k e^{\lambda'_k t_n}[\sin(\lambda''_k t_n) \sin(k(x_m - x_0)) \\
&+ \cos(\lambda''_k t_n) \cos(k(x_m - x_0))] \\
&+ \sum_k e^{\lambda'_k (t_n+\tau)}[\sin(\lambda''_k (t_n + \tau)) \sin(k(x_m - x_0)) \\
&+ \cos(\lambda''_k (t_n + \tau)) \cos(k(x_m - x_0))]\Big)
\end{aligned}
$$

Fortsetzung

$$k = \frac{2\pi l}{L} \qquad l = 0, 1 ... M - 1$$

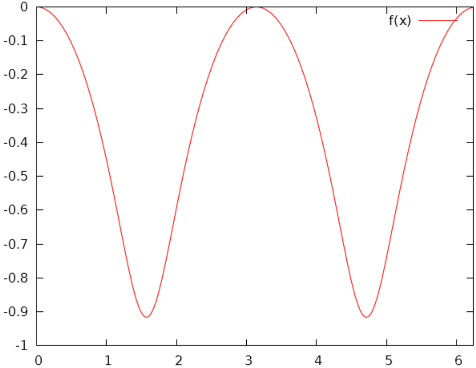
$$\cos(x \pm y) = \sin(x)\sin(y) \pm \cos(x)\cos(y)$$

$$\overline{P}(m,n) = \frac{1}{2M}\Big(\sum_{l=0}^{M-1} e^{\lambda'_k n}\cos(\lambda''_k n + \frac{2\pi l}{M}m) + e^{\lambda'_k (n+1)}\cos(\lambda''_k (n+1)\frac{2\pi l}{M}m)\Big)$$

Übergang $n \to \infty$

$$e^{\lambda'_k n} \to 0 \qquad \text{für} \qquad \lambda'_k < 0$$

$$\lambda'_k = \frac{1}{\tau} \ln(\sqrt{\cos k^2 + \Delta^2 \sin k})$$



$$\lambda'_k = 0 \text{ für } k = 0, \pi$$

$$k = \frac{2\pi l}{L} = \pi$$

$$\to \quad l = \frac{L}{2}$$

Fortsetzung

$$\lambda_k'' = \frac{1}{\tau} \arctan(\Delta \tan) = 0 \text{ für } k = 0, \pi$$

$$\overline{P}(m) = \frac{1}{M}(\cos(0) + \cos(\pi m))$$

Mastergleichung

$$P(x_m, t) = \sum_k e^{\lambda'_k t}[\sin(\lambda''_k t)\sin(k(x_m - x_0)) + \cos(\lambda''_k t)\cos(k(x_m - x_0))]$$

$$\lambda'_k = (w_+ + w_-)(1 - cos(k))$$

$$\lambda''_k = ((w_+ - w_-)sin(k)$$

Übergang $n \to \infty$

$$P(m,t) = \sum_k e^{\lambda'_k} \cos(\lambda''_k t + km)$$

$$w_+ + w_- = \frac{p}{\tau} + \frac{q}{\tau} = 1$$

$$\lambda'_k = (w_+ + w_-)(1 - cos(k)) = 0 \text{ für } = 0,$$

$$\lambda''_k = ((w_+ - w_-)sin(k) = \Delta \sin(k) = 0 \text{ für } k = \pi$$

$$P(m) = \frac{1}{M} \cos(0)$$