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Problem Set 9 (due Monday, 14.12.2015 in the lecture)

Questions

(Q1) A second-quantized many-body Hamiltonian for some given interaction looks the same for fermions and bosons. The physical phenomena, however, may be very different. How come?

(Q2) What are Phonons and why are they bosonic?

(Q3) What does it mean when we say that a relativistically correct wave equation should be "Lorentz covariant"?

(9.1) Bosonic creation and annihilation operators *(3 points)*

Unlike in the lecture, now start with the bosonic commutator relation

$$[\hat{b}, \hat{b}^\dagger] = \hat{1}$$

and the occupation number operator

$$\hat{n} = \hat{b}^\dagger \hat{b}, \qquad \hat{n}\,|n\rangle = n\,|n\rangle$$

to show that

(i) $[\hat{b}^q, \hat{n}] = q\hat{b}^q$ and $[\hat{b}^{\dagger q}, \hat{n}] = -q\hat{b}^{\dagger q}$, with q a positive integer, and

(ii) $\hat{b}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{b}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

(9.2) Some fun facts *(3 points)*

Let $\hat{b}$ and $\hat{a}$ be bosonic and fermionic annihilation operators, respectively. Show that both

$$\mathrm{e}^{\alpha\hat{b}^\dagger\hat{b}}\hat{b}\,\mathrm{e}^{-\alpha\hat{b}^\dagger\hat{b}} = \mathrm{e}^{-\alpha}\hat{b}, \qquad \mathrm{e}^{\alpha\hat{a}^\dagger\hat{a}}\hat{a}\,\mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}} = \mathrm{e}^{-\alpha}\hat{a}$$

where $\alpha \in \mathbb{C}$.

* Show that $\mathrm{e}^{\alpha\hat{b}^\dagger\hat{b}}\hat{b}^\dagger\,\mathrm{e}^{-\alpha\hat{b}^\dagger\hat{b}} = \mathrm{e}^{\alpha}\hat{b}^\dagger$, and the same for fermions.

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(9.3) STIMULATED EMISSION OF PHONONS *(3 points)*

In the lecture, we started from an initial state with just one electron, $|\Psi(0)\rangle = \hat{a}^\dagger_{k_0}|0\rangle$, and found for the spontaneous emission of a phonon in first order perturbation theory $\left|\Psi'_{\text{spon}}(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\sum_\kappa g^*_\kappa \int_0^t \mathrm{e}^{\mathrm{i}(\Omega_{k_0-\kappa}+\omega_\kappa-\Omega_{k_0})\tau}\mathrm{d}\tau\, \hat{a}^\dagger_{k_0-\kappa}\hat{b}^\dagger_\kappa|0\rangle + \hat{a}^\dagger_{k_0}|0\rangle$ (look up the lecture notes in case you do not understand the notation).

Perform a similar calculation for the process of *stimulated* emission of phonons. Assume an initial state with n phonons in mode κ_0 and one electron in state k_0,

$$|\Psi(0)\rangle = \hat{a}^\dagger_{k_0}\frac{(\hat{b}^\dagger_{\kappa_0})^n}{\sqrt{n!}}|0\rangle\,, \tag{1}$$

and show that

$$\left|\Psi'_{\text{stim}}(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}g^*_{\kappa_0}\int_0^t \mathrm{e}^{\mathrm{i}(\Omega_k+\omega_{\kappa_0}-\Omega_{k+\kappa_0})\tau}\mathrm{d}\tau\,\delta_{k+\kappa_0,k_0}\sqrt{n+1}\,\hat{a}^\dagger_k\frac{(\hat{b}^\dagger_{\kappa_0})^{n+1}}{\sqrt{(n+1)!}}|0\rangle + |\Psi(0)\rangle\,.$$

* Which other first-order processes are possible, given the initial state (1)?

(9.4) REPETITION: TENSOR CALCULUS *(1 point)*

Consider a coordinate transformation $x \to \bar{x} = f(x)$ and the inverse $\bar{x} \to x = h(\bar{x})$ (here, we collect all $x^1, x^2, \ldots$ in x, and all $\bar{x}^1, \bar{x}^2, \ldots$ in $\bar{x}$).

Per definition, a *scalar field* transforms according $\bar{\varphi}(\bar{x}) = \varphi(x)$, i.e., for the same point represented by x in one coordinate system and $\bar{x}$ in the other, $\varphi(x)$ and $\bar{\varphi}(\bar{x})$ have the same numerical value.

The differentials $\mathrm{d}\bar{x}^\mu$ transform according $\mathrm{d}\bar{x}^\mu = \frac{\partial f^\mu}{\partial x^\nu}\mathrm{d}x^\nu$ (summation over ν), which we write as

$$\mathrm{d}\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}\mathrm{d}x^\nu. \tag{2}$$

By definition, any n-tuple A^μ which transforms like (2),

$$\bar{A}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}A^\nu, \tag{3}$$

is a *contravariant vector*.

Instead, the partial derivatives of a scalar field transform according

$$\frac{\partial\bar{\varphi}(\bar{x})}{\partial\bar{x}^\mu} = \frac{\partial\varphi(x)}{\partial\bar{x}^\mu} = \frac{\partial x^\nu}{\partial\bar{x}^\mu}\frac{\partial\varphi(x)}{\partial x^\nu}. \tag{4}$$

By definition, any n-tuple B_μ which transforms like (4),

$$\bar{B}_\mu = \frac{\partial x^\nu}{\partial\bar{x}^\mu}B_\nu, \tag{5}$$

is a *covariant vector*.

Show that $\bar{S} = \bar{A}^\mu\bar{B}_\mu = A^\mu B_\mu = S$, i.e., $A^\mu B_\mu$ is a scalar.