Prof. Dr. Dieter Bauer
Dr. Yaroslav Lutsyshyn
Dipl.-Phys. Thomas Keil

Advanced Quantum Theory
WS 13/14

Problem Set 9 (due Monday, 16.12.2013 in the lecture)

Questions

(Q1) What is the advantage of writing everything in terms of creation and annihilation operators?

(Q2) What is ferromagnetism and what has the Heisenberg exchange operator to do with it?

(9.1) Two-particle interaction in second quantization *(4 points)*

Let

$$\hat{W} = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \hat{w}_{ij}$$

be the interaction part in a many-body Hamiltonian (e.g., the Coulomb interaction $\hat{w}_{ij} = e^2/|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j|$). Show (by a calculation similar to the one in section 4.5.1 of the lecture notes) that

$$\hat{W} = \frac{1}{2} \sum_{k_1 k_2 k_1' k_2'} \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} w(k_1 k_2, k_1' k_2') \hat{a}_{k_2'} \hat{a}_{k_1'} \quad , \qquad w(k_1 k_2, k_1' k_2') = \langle k_1 k_2 | \hat{w}_{12} | k_1' k_2' \rangle.$$

(9.2) Constructing a spin algebra *(3 points)*

$\hat{a}^\dagger_\pm$ and $\hat{a}_\pm$ create and annihilate a Fermion of spin 1/2 at a given lattice site, respectively. Using (anti-) commutator relations, show that $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$ defined via

$$\hat{s}_\pm = \hat{s}_x \pm \mathrm{i}\hat{s}_y = \hbar \hat{a}^\dagger_\pm \hat{a}_\mp$$

together with

$$\hat{s}_z = \frac{\hbar}{2}(\hat{n}_+ - \hat{n}_-), \qquad \hat{n}_\pm = \hat{a}^\dagger_\pm \hat{a}_\pm$$

satisfy indeed the angular momentum relations

$$[\hat{s}_x, \hat{s}_y] = -\frac{\hbar}{\mathrm{i}} \hat{s}_z \qquad \text{(and cyclic)},$$

as claimed in the lecture.

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(9.3) Heisenberg exchange operator *(3 points)*

Show that

$$\hat{H}^{\mathrm{A}}_{\mathrm{int}} = \frac{1}{2} \sum_{n_1 \neq n_2} \sum_{m_1 m_2} \hat{a}^{\dagger}_{n_1 m_1} \hat{a}^{\dagger}_{n_2 m_2} K_{n_1 n_2} \hat{a}_{n_1 m_2} \hat{a}_{n_2 m_1}$$

can indeed be written as

$$\hat{H}^{\mathrm{A}}_{\mathrm{int}} = - \sum_{n_1 \neq n_2} K_{n_1 n_2} \left(\frac{1}{\hbar^2} \hat{\mathbf{s}}_{n_1} \cdot \hat{\mathbf{s}}_{n_2} + \frac{1}{4} \hat{1} \right),$$

as claimed in the lecture.