Advanced Quantum Theory
WS 11/12

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Problem Set 9 (due 13.12.2011)

Questions

(Q1) What is the advantage of introducing creation and annihilation operators? In other words: why do we need a *second quantization*?

(Q2) Why is it called *second quantization*?

(Q3) What is (anti-) ferromagnetism and what has the Heisenberg exchange operator [calculated below in (9.3)] to do with it?

(9.1) Repetition: Bosonic creation and annihilation operators *(3 points)*

In the lecture we mostly dealt with fermionic systems so far. Now let $\hat{b}$ and $\hat{b}^\dagger$ be bosonic annihilation and creation operators,

$$[\hat{b}, \hat{b}^\dagger] = \hat{1}$$

and

$$\hat{n} = \hat{b}^\dagger \hat{b}$$

the occupation number operator, $\hat{n}|n\rangle = n|n\rangle$. Show that

(i) $[\hat{b}^q, \hat{n}] = q\hat{b}^q$ and $[\hat{b}^{\dagger q}, \hat{n}] = -q\hat{b}^{\dagger q}$, with q a positive integer, and
(ii) $\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{b}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(9.2) Constructing a spin algebra with annihilation and creation operators *(3 points)*

$\hat{a}^\dagger_\pm$ and $\hat{a}_\pm$ create and annihilate a Fermion of spin $1/2$ at a given lattice site, respectively. Using (anti-) commutator relations, show that $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$ defined via

$$\hat{s}_\pm = \hat{s}_x \pm \mathrm{i}\hat{s}_y = \hbar \hat{a}^\dagger_\pm \hat{a}_\mp$$

together with

$$\hat{s}_z = \frac{\hbar}{2}(\hat{n}_+ - \hat{n}_-), \qquad \hat{n}_\pm = \hat{a}^\dagger_\pm \hat{a}_\pm$$

satisfy indeed the angular momentum relations

$$[\hat{s}_x, \hat{s}_y] = -\frac{\hbar}{\mathrm{i}}\hat{s}_z \qquad \text{(and cyclic)},$$

as claimed in the lecture.

(9.3) Heisenberg exchange operator *(4 points)*

Show that

$$\hat{H}^{\mathrm{A}}_{\mathrm{int}} = \frac{1}{2}\sum_{n_1 \neq n_2}\sum_{m_1 m_2} \hat{a}^\dagger_{n_1 m_1}\hat{a}^\dagger_{n_2 m_2} K_{n_1 n_2} \hat{a}_{n_1 m_2}\hat{a}_{n_2 m_1}$$

can indeed be written as

$$\hat{H}^{\mathrm{A}}_{\mathrm{int}} = -\sum_{n_1 \neq n_2} K_{n_1 n_2}\left(\frac{1}{\hbar^2}\hat{\mathbf{s}}_{n_1} \cdot \hat{\mathbf{s}}_{n_2} + \frac{1}{4}\hat{1}\right),$$

as claimed in the lecture.