Advanced Quantum Theory
WS 12/13

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Problem Set 8 (due 10.12.2012 in the lecture)

Questions

(Q1) What is the advantage of introducing creation and annihilation operators?

(Q2) What is ferromagnetism and what has the Heisenberg exchange operator to do with it?

(8.1) Two-particle time-dependent Schrödinger equation *(4 points)*

Consider a fermionic Hamiltonian in second quantization

$$\hat{H} = \sum_{kk'} \varepsilon(k,k')\, \hat{a}_k^\dagger \hat{a}_{k'}, \qquad \varepsilon(k,k') = \langle k| \left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{r}}) \right) |k'\rangle.$$

Modify the procedure outlined in Section 4.5.3 of the lecture notes to derive the two-particle time-dependent Schrödinger equation in "first quantization" and position-spin representation

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \Phi^-_{m_1 m_2}(\mathbf{r}_1, \mathbf{r}_2; t) = \sum_{i=1}^{2} \left(-\frac{\hbar^2}{2m} \nabla_i^2 + V(\mathbf{r}_i) \right) \Phi^-_{m_1 m_2}(\mathbf{r}_1, \mathbf{r}_2; t)$$

where $\Phi^-_{m_1 m_2}(\mathbf{r}_1, \mathbf{r}_2; t)$ is a two-particle, antisymmetrized wavefunction.

(8.2) Constructing a spin algebra with annihilation and creation operators *(3 points)*

$\hat{a}_\pm^\dagger$ and $\hat{a}_\pm$ create and annihilate a Fermion of spin $1/2$ at a given lattice site, respectively. Using (anti-) commutator relations, show that $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$ defined via

$$\hat{s}_\pm = \hat{s}_x \pm \mathrm{i}\hat{s}_y = \hbar \hat{a}_\pm^\dagger \hat{a}_\mp$$

together with

$$\hat{s}_z = \frac{\hbar}{2}(\hat{n}_+ - \hat{n}_-), \qquad \hat{n}_\pm = \hat{a}_\pm^\dagger \hat{a}_\pm$$

satisfy indeed the angular momentum relations

$$[\hat{s}_x, \hat{s}_y] = -\frac{\hbar}{\mathrm{i}} \hat{s}_z \qquad \text{(and cyclic)},$$

as claimed in the lecture.

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(8.3) Heisenberg exchange operator *(3 points)*

Show that

$$\hat{H}_{\text{int}}^{\text{A}} = \frac{1}{2} \sum_{n_1 \neq n_2} \sum_{m_1 m_2} \hat{a}_{n_1 m_1}^\dagger \hat{a}_{n_2 m_2}^\dagger K_{n_1 n_2} \hat{a}_{n_1 m_2} \hat{a}_{n_2 m_1}$$

can indeed be written as

$$\hat{H}_{\text{int}}^{\text{A}} = - \sum_{n_1 \neq n_2} K_{n_1 n_2} \left(\frac{1}{\hbar^2} \hat{\mathbf{s}}_{n_1} \cdot \hat{\mathbf{s}}_{n_2} + \frac{1}{4} \hat{1} \right),$$

as claimed in the lecture.