Advanced Quantum Theory
WS 10/11

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Problem Set 8 (due 09.12.2010)

(8.1) Transformation of annihilation and creation operators *(3 points)*

Show that the annihilation operators $\hat{\psi}_m(\mathbf{r})$ and $\hat{a}_k$ obey [cf. (V17) in the lecture notes]

$$\hat{\psi}_m(\mathbf{r}) = \sum_k \hat{a}_k \langle \mathbf{r}m|k\rangle.$$

(8.2) Commutators and anti-commutators *(3 points)*

Show that for arbitrary operators $\hat{A}$, $\hat{B}$, $\hat{C}$

$$[\hat{A}\hat{B}, \hat{C}]_- = \hat{A}[\hat{B}, \hat{C}]_\mp \pm [\hat{A}, \hat{C}]_\mp \, \hat{B},$$

$$[\hat{A}\hat{B}, \hat{C}]_+ = \hat{A}[\hat{B}, \hat{C}]_\pm \mp [\hat{A}, \hat{C}]_\mp \, \hat{B}$$

where on the right hand sides one can either choose the upper signs or the lower signs.

(8.3) Constructing a spin algebra with annihilation and creation operators *(4 points)*

$\hat{a}^\dagger_\pm$ and $\hat{a}_\pm$ create and annihilate a Fermion of spin $1/2$ at a given lattice site, respectively. Using (anti-) commutator relations, show that $\hat{s}_x$, $\hat{s}_y$, $\hat{s}_z$ defined via

$$\hat{s}_\pm = \hat{s}_x \pm \mathrm{i}\hat{s}_y = \hbar \hat{a}^\dagger_\pm \hat{a}_\mp$$

together with

$$\hat{s}_z = \frac{\hbar}{2}(\hat{n}_+ - \hat{n}_-), \qquad \hat{n}_\pm = \hat{a}^\dagger_\pm \hat{a}_\pm$$

satisfy indeed the angular momentum relations

$$[\hat{s}_x, \hat{s}_y] = -\frac{\hbar}{\mathrm{i}}\hat{s}_z \qquad \text{(and cyclic)},$$

as claimed in the lecture.