Prof. Dr. Dieter Bauer
Dr. Yaroslav Lutsyshyn
M.Sc. Julius Rapp

Problem Set 7 (due Monday, 30.11.2015 in the lecture)

Questions

(Q1) What is the advantage of writing everything in terms of creation and annihilation operators?

(Q2) What are the dimensions of creation and annihilation operators?

(7.1) Fermionic annihilation operator *(2 points)*

Using

$$\hat{1} = |0\rangle\langle 0| + \sum_{k_1} |k_1\rangle\langle k_1| + \sum_{k_1<k_2} |k_1 k_2^-\rangle\langle k_1 k_2^-| + \cdots ,$$

and

$$\hat{a}_k^\dagger |k_1 \ldots k_N^-\rangle = |k k_1 \ldots k_N^-\rangle,$$

show that

$$\hat{a}_k |k_1 \ldots k_N^-\rangle = \delta_{kk_1} |k_2 \ldots k_N^-\rangle - \delta_{kk_2} |k_1 k_3 \ldots k_N^-\rangle + \cdots - \cdots .$$

(7.2) Transformations *(2 points)*

Show that the annihilation operators $\hat{\psi}_m(\boldsymbol{r})$ and $\hat{a}_k$ obey

$$\hat{\psi}_m(\boldsymbol{r}) = \sum_k \hat{a}_k \langle \boldsymbol{r} m | k \rangle.$$

How do the transformation for the respective creation operators and the back-transformations read?

cont'd overleaf

(7.3) TWO-PARTICLE INTERACTION IN SECOND QUANTIZATION *(3 points)*

Let

$$\hat{W} = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \hat{w}_{ij}$$

be the interaction part in a many-body Hamiltonian (e.g., the Coulomb interaction $\hat{w}_{ij} = e^2/|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j|$). Show (by a calculation similar to the one in section 4.5.1 of the lecture notes) that "in second-quantized form"

$$\hat{W} = \frac{1}{2} \sum_{k_1 k_2 k_1' k_2'} \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} w(k_1 k_2, k_1' k_2') \hat{a}_{k_2'} \hat{a}_{k_1'} \quad , \qquad w(k_1 k_2, k_1' k_2') = \langle k_1 k_2 | \hat{w}_{12} | k_1' k_2' \rangle.$$

(7.4) CONSERVATION OF TOTAL PARTICLE NUMBER *(3 points)*

Show that for a second-quantized Hamiltonian of the form

$$\hat{H} = \sum_{k_1 k_2} \hat{a}^\dagger_{k_1} v(k_1, k_2) \hat{a}_{k_2} + \frac{1}{2} \sum_{k_1 k_2 k_1' k_2'} \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} w(k_1 k_2, k_1' k_2') \hat{a}_{k_2'} \hat{a}_{k_1'}$$

the total number of particles $\hat{n} = \sum_k \hat{n}_k = \sum_k \hat{a}^\dagger_k \hat{a}_k$ is conserved, i.e.,

$$[\hat{H}, \hat{n}] = 0.$$

Remark: Although this is true for both bosons and fermions, show it for fermions.

*Which kind of terms in the Hamiltonian would be required to *not* conserve the total particle number?