Advanced Quantum Theory
WS 10/11

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Problem Set 5 (due 18.11.2010)

(5.1) A carbon configuration *(5 points)*

Let us consider the carbon configuration

$$(1s)^2\ (2s)^2\ (2p)\ (3d).$$

(i) Show that the possible terms ${}^{2S+1}L$ are

$$^1P,\ ^3P,\ ^1D,\ ^3D,\ ^1F,\ ^3F$$

and determine their degeneracies.

(ii) Determine the fine-structure multiplet ${}^{2S+1}L_J$ for the ${}^3P$-term above and specify the degeneracy of each fine-structure term.

(5.2) Spin-orbit coupling in H *(5 points)*

Consider hydrogenic states $|n\ell m_\ell s m_s\rangle$. With spin-orbit coupling $\hat{\boldsymbol{\ell}} \cdot \hat{\boldsymbol{s}}$ present, the good quantum numbers are n, j, m_j, ℓ, s, with $\hat{\boldsymbol{j}} = \hat{\boldsymbol{\ell}} + \hat{\boldsymbol{s}}$. For a given n one may switch from the uncoupled basis $|\ell m_\ell s m_s\rangle$ to the coupled basis $|j m_j \ell s\rangle$,

$$|j m_j \ell s\rangle = \sum_{m_\ell} \sum_{m_s} |\ell m_\ell s m_s\rangle \langle \ell m_\ell s m_s | j m_j\rangle.$$

Using spherical harmonics $Y_\ell^m(\Omega)$ and spinors $\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ this change of basis may be written as

$$\mathcal{Y}_{j\ell}^{m_j}(\Omega) := \langle \Omega | j m_j \ell s\rangle = \sum_{m_\ell} \sum_{m_s} Y_\ell^m(\Omega) \chi_{m_s} \langle \ell m_\ell s m_s | j m_j\rangle$$

where the spinors $\mathcal{Y}_{j\ell}^{m_j}(\Omega)$ are called *generalized spherical harmonics*, and $\langle \ell m_\ell s m_s | j m_j\rangle$ are Clebsch-Gordan coefficients.[1]

Determine $\mathcal{Y}_{j\ell}^{m_j}(\Omega)$, show that it is an eigenfunction of the operator $\hat{\boldsymbol{\ell}} \cdot \hat{\boldsymbol{s}}$, and calculate the eigenvalue.

[1] In general, the Clebsch-Gordan coefficient $\langle a\alpha\frac{1}{2}\beta|c\gamma\rangle$ is non-vanishing only for $\alpha + \beta = \gamma$. For $c = a + \frac{1}{2}$ and $\beta = \pm\frac{1}{2}$ it is given by $\left(\frac{c \pm \gamma}{2c}\right)^{1/2}$. For $c = a - \frac{1}{2}$ and $\beta = \pm\frac{1}{2}$ it is given by $\mp\left(\frac{c \mp \gamma + 1}{2c+2}\right)^{1/2}$.