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Problem Set 13 (due Monday, 25.01.2016 in the lecture)

Questions

(Q1) Now, in Chapter 6, we arrived at the bosonic commutator relation $[b_{\mathbf{k}}, b^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}$ via field quantization. The same commutator relation was obtained in Chapter 4 by different means. Where's the connection?

(Q2) How do anti-particles emerge in the formalism of field quantization?

(Q3) In Chapter 5, we abandoned the Klein-Gordon equation because of a non-positive definite probability density. Why isn't that a problem anymore after second quantization in Chapter 6?

(13.1) Real Klein-Gordon field: energy and momentum *(3 points)*

Starting from

$$H = \int \mathrm{d}^3 r \, \frac{1}{2} \left\{ \frac{1}{c^2} (\dot{\phi})^2 + (\nabla \phi)^2 + \mu^2 \phi^2 \right\}, \qquad \mathbf{P} = - \int \mathrm{d}^3 r \, \frac{1}{c^2} \dot{\phi} \nabla \phi,$$

derive

$$H = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left(b^{\dagger}(\mathbf{k}) b(\mathbf{k}) + \frac{1}{2} \right), \qquad \mathbf{P} = \sum_{\mathbf{k}} \hbar \mathbf{k} \left(b^{\dagger}(\mathbf{k}) b(\mathbf{k}) + \frac{1}{2} \right)$$

using the expansion of the real Klein-Gordon field $\phi(x)$ in bs and $b^{\dagger}$s in section 6.4 of the lecture notes.

(13.2) Covariant equation of motion *(2 points)*

Show that for some function $F(x) = F(\phi_r(x), \pi_r(x))$, which can be expanded in $\phi_r(x)$ and $\pi_r(x)$,

$$[P^j, F(x)] = -\mathrm{i}\hbar \frac{\partial F(x)}{\partial x_j}, \qquad j = 1, 2, 3 \tag{1}$$

holds.[1]

cont'd overleaf

[1] Note that combining (1) with the Heisenberg equation of motion $[H, F(x)] = -\mathrm{i}\hbar c \frac{\partial F(x)}{\partial x_0}$ and $x^0 = x_0 = ct$, $P^0 = H/c$ one obtains the *covariant equation of motion*

$$[P^{\alpha}, F(x)] = -\mathrm{i}\hbar \frac{\partial F(x)}{\partial x_{\alpha}}, \quad \alpha = 0, 1, 2, 3.$$

(13.3) COVARIANT COMMUTATOR RELATION *(2 points)*

Show that with the covariant commutator relation for the real Klein-Gordon equation introduced in the lecture,

$$[\phi(x), \phi(y)] = \mathrm{i}\hbar c \Delta(x-y) \tag{2}$$

with

$$\Delta(x) = -\frac{c}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{\omega_{\mathbf{k}}} \sin kx, \qquad \omega_{\mathbf{k}}^2 = c^2(\mu^2 + \mathbf{k}^2),$$

the equal-time commutator relation used for quantization

$$[\phi(\mathbf{r}, t), \dot{\phi}(\mathbf{r}', t)] = \mathrm{i}\hbar c^2 \delta(\mathbf{r} - \mathbf{r}')$$

is "automatically" fulfilled.

Hint: Take the derivative $\dfrac{\partial}{\partial y^0} = \dfrac{\partial}{\partial ct'}$ of (2).

(13.4) LAGRANGE DENSITY OF THE FREE DIRAC FIELD *(3 points)*

By applying the Euler-Lagrange equation with respect to $\psi_m^\dagger(x)$ and $\psi_m(x)$, $m = 1, 2, 3, 4$, to the Lagrange density

$$\mathcal{L} = c\overline{\psi}(x)(\mathrm{i}\hbar\gamma^\mu\partial_\mu - mc)\psi(x) \tag{3}$$

with

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0,$$

show that the free Dirac equation

$$(\mathrm{i}\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0$$

and its adjoint

$$\psi^\dagger(x)(\mathrm{i}\hbar\overleftarrow{\partial}_\mu\gamma^{\mu\dagger} + mc) = 0$$

result, respectively.

Remark: Note that there are actually summations over bispinor indices r, s hidden in (3):

$$\mathcal{L} = c\psi_r^\dagger(x)\left(\mathrm{i}\hbar\gamma^0\gamma^\mu\partial_\mu - mc\gamma^0\right)_{rs}\psi_s(x).$$