Prof. Dr. Dieter Bauer
Dr. Yaroslav Lutsyshyn
M.Sc. Julius Rapp

Problem Set 12 (due Monday, 18.01.2016 in the lecture)

Questions

(Q1) What is the Klein paradox?

(Q2) What is the Noether Theorem and why is it of so immense importance?

(12.1) Lagrange density *(3 points)*

Show that the Lagrange density

$$\mathcal{L}' = \mathcal{L} + \partial_\alpha \Lambda^\alpha$$

with Λ^α arbitrary functions of the fields $\phi_r(x)$ yields the same field equations as $\mathcal{L}$ does.

(12.2) Klein-Gordon Hamilton density *(4 points)*

(i) Using the Lagrange density $\mathcal{L}$ and conjugate field π derived in the lecture, show that the Hamilton density of the Klein-Gordon field reads

$$\mathcal{H} = \frac{1}{2}\left(c^2\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2\right).$$

(ii) Show that with $H = \int \mathrm{d}^3r\,\mathcal{H}$ and the commutator relations (introduced in the lecture) follows

$$[H, \phi(x)] = -\mathrm{i}\hbar c^2 \pi(x), \qquad [H, \pi(x)] = \mathrm{i}\hbar(\mu^2 - \nabla^2)\phi(x).$$

(iii) Show that from (ii) and the Heisenberg equations of motion for $\phi(x)$ and $\pi(x)$

$$\dot{\phi}(x) = c^2\pi(x), \qquad (\Box + \mu^2)\phi(x) = 0$$

follow.

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(12.3) Generalized Euler-Lagrange equation *(3 points)*

One might wonder what happens if we allow for Lagrangians containing higher-order derivatives,

$$\mathcal{L} = \mathcal{L}(\phi,\ \partial_\alpha\phi,\ \partial_\alpha\partial_\beta\phi,\ \partial_\alpha\partial_\beta\partial_\gamma\phi,\ \ldots) \tag{1}$$

(for simplicity, we restrict ourselves to one-component, real fields only; the generalization to ϕ_r, $r = 1, 2, \ldots N$ is straightforward).

In fact, such Lagrangians occur in general relativity, in string theory, or when corrections (i.e., Taylor expansions of potentials or interactions) are taken into account. They may lead to unpleasant (and sometimes to disastrous) consequences though.

(i) Derive the Euler-Lagrange equation for a Lagrangian of the form (1).
Hint: Follow the derivation of the "standard" Euler-Lagrange equation in the lecture notes, and just extend it in the obvious way. You may assume that all variations on the boundary $\Gamma(\Omega)$ vanish.

*(ii) Derive the classical equation of motion from the Lagrangian

$$L(q, \dot{q}, \ddot{q}) = \frac{1}{2}(1 + \epsilon^2\omega^2)\dot{q}^2 - \frac{1}{2}\omega^2 q^2 - \frac{1}{2}\epsilon^2\ddot{q}^2, \qquad \omega^2, \epsilon^2 \geq 0$$

and find a solution for $q(t)$. How many initial conditions do you need? Do you obtain the harmonic oscillator solution for $\epsilon \to 0$?