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Problem Set 11 (due Monday, 11.01.2016 in the lecture)

Questions

(Q1) How does a bispinor transform under Lorentz transformations? Describe verbally how we derived it.

(Q2) What is the minimum energy $\hbar\omega$ (in eV) of a photon necessary to create an electron-positron pair?

(Q3) What is wrong with the following argument? "If $\hbar\omega$ is too small to create an electron-positron pair in the lab frame I change to a counter-propagating reference frame such that—due to the relativistic Doppler effect—$\hbar\omega'$ is large enough, and an electron-positron pair can be created."

(11.1) Hermitized Darwin term for Coulomb potential *(3 points)*

Show that the hermitized Darwin term (see lecture notes) for a Coulomb potential $V(r) = -Ze^2/r$ (cgs units) reads

$$\hat{H}_{\mathrm{D}}^{\mathrm{h}} = \frac{\pi\hbar^2 Ze^2}{2m^2c^2}\delta(\mathbf{r}).$$

* Why does this affect s-states only?

(11.2) Four-dimensional spin matrices *(3 points)*

Show (without using a particular representation of the Dirac γ-matrices but only (anti-) commutator relations) that the four-dimensional spin matrices

$$\sigma_{\alpha\mu} = \frac{\mathrm{i}}{2}[\gamma_\alpha, \gamma_\mu]$$

indeed fulfill

$$2\mathrm{i}(g_\alpha^\nu\gamma_\mu - g_\mu^\nu\gamma_\alpha) = [\gamma^\nu, \sigma_{\alpha\mu}],$$

as claimed in the lecture.

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(11.3) ADJOINT GAMMA MATRICES *(2 points)*

Show that

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0.$$

Hint: Remember that the "original" matrices α_i and β are hermitian.

(11.4) RELATIVISTIC PROBABILITY DENSITY *(2 points)*

The bispinor of a particle at rest reads (cf. lecture notes)

$$\psi_r^{(0)}(x) = w_r(\mathbf{0})\, \mathrm{e}^{-\mathrm{i}\epsilon_r mc^2 t/\hbar}, \qquad r = 1, 2, 3, 4$$

with $\epsilon_r = 1$ for $r = 1, 2$ and -1 for $r = 3, 4$. For finite momentum we derived

$$\psi_r(x) = w_r(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\epsilon_r p_\mu x^\mu/\hbar}.$$

Let us consider $r = 1$, i.e., a positive-energy, spin-up electron for which

$$w_1(\mathbf{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_1(\mathbf{p}) = \sqrt{\frac{\mathcal{E}}{2mc^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{\mathcal{E}} \\ \frac{p_+ c}{\mathcal{E}} \end{pmatrix}, \quad \mathcal{E} = E + mc^2, \quad p_\pm = p_x \pm \mathrm{i} p_y.$$

Show that the probability density $\rho(x) = \psi_1^\dagger(x)\psi_1(x)$ is by a factor γ higher than $\rho^{(0)}(x) = \psi_1^{(0)\dagger}(x)\psi_1^{(0)}(x)$, as it should.

* Why should it?