Advanced Quantum Theory
WS 12/13

Prof. Dr. Dieter Bauer
Dr. Yaroslav Lutsyshyn
Dipl.-Phys. Volker Mosert

Problem Set 11 (due 14.01.2013 in the lecture)

Questions

(Q1) What is the g-factor (aka “gyromagnetic factor” or “Landé factor”)?

(Q2) How does a Dirac bispinor transform under Lorentz transformations? Describe verbally how we derived it.

(Q3) What is a proper orthochronous (in German *eigentliche*) Lorentz transformation, what are the others?

(11.1) Adjoint gamma matrices *(3 points)*

Show that

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0.$$

Hint: Remember that the “original” matrices α_i and β are hermitian.

(11.2) Bispinor transformation for spatial reflection, and parity *(3 points)*

The Lorentz transformation describing spatial reflection reads

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Show that in this case the corresponding bispinor transformation $\psi'(x') = S(\Lambda)\psi(x)$ is simply given by

$$S = \mathrm{e}^{\mathrm{i}\varphi}\gamma^0$$

with $\mathrm{e}^{\mathrm{i}\varphi}$ an arbitrary phase factor.

Note: The full bispinor *parity operator* is accordingly defined as $\hat{\mathcal{P}} = S\hat{\mathcal{P}}^{(0)}$ where $\hat{\mathcal{P}}^{(0)}$ performs a spatial reflection $\mathbf{r} \to -\mathbf{r}$.

The eigenstates of the free Dirac equation in the particle’s rest frame (see lecture notes, beginning of Sec. 5.1.2) are also eigenstates of γ^0 but with opposite eigenvalues for the particle-like and antiparticle-like solutions. As a consequence, the *intrinsic parities* (i.e., those due to spin) are opposite.

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(11.3) Four-dimensional spin matrices *(4 points)*

Show (without using a particular representation of the Dirac γ-matrices but only (anti-) commutator relations) that the four-dimensional spin matrices

$$\sigma_{\alpha\mu} = \frac{\mathrm{i}}{2}[\gamma_\alpha, \gamma_\mu]$$

indeed fulfill

$$2\mathrm{i}(g^\nu_\alpha \gamma_\mu - g^\nu_\mu \gamma_\alpha) = [\gamma^\nu, \sigma_{\alpha\mu}],$$

as claimed in the lecture.

(11.4) *Relativistic total angular momentum conservation of a Dirac particle

We can write the relativistic spin component Σ_i, $i = 1, 2, 3$ of a Dirac particle [using the spin matrices from (11.3)] in the form

$$\Sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk}.$$

For the case $i = 3$ discussed in the lecture (i.e., rotations about the z-axis) we recover indeed $\Sigma_3 = (\epsilon_{312}\sigma^{12} + \epsilon_{321}\sigma^{21})/2 = (\sigma^{12} - \sigma^{21})/2 = (\sigma^{12} + \sigma^{12})/2 = \sigma^{12}$.

Now, let

$$\hat{\mathbf{J}} = \mathbf{r} \times \hat{\mathbf{p}} \otimes \hat{1} + \frac{\hbar}{2}\mathbf{\Sigma}$$

with $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ be the relativistic total angular momentum.

Show that $\hat{\mathbf{J}}$ commutes with the free Dirac Hamiltonian $\hat{H} = c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta mc^2$,

$$[\hat{\mathbf{J}}, \hat{H}] = 0,$$

which means that the relativistic total angular momentum is conserved.

Note: Not surprisingly, the same is true for $\hat{H}' = \hat{H} + e\phi(r)$, i.e., for a Dirac particle in a scalar central-field potential.