Advanced Quantum Theory

WS 12/13

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Problem Set 10 (due 07.01.2013 in the lecture)

Questions

(Q1) What does it mean when we say that a relativistically correct wave equation should be "Lorentz covariant"?

(Q2) The Dirac equation is a wave equation for a four-component bispinor ψ. Why four?

(10.1) Pauli matrices *(3 points)*

Show that $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\,1 + \mathrm{i}\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$. Here, $\mathbf{a}$ and $\mathbf{b}$ are arbitrary three-vectors and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices.

(10.2) Dirac matrices *(4 points)*

(i) Show explicitly that the 4×4 matrices

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1,2,3, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(with σ_i the Pauli matrices) fulfill the anticommutator relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \qquad \alpha_i\beta + \beta\alpha_i = 0, \qquad \beta^2 = 1.$$

(ii) Show that the 4×4 *gamma matrices* $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$, $i = 1,2,3$ fulfill

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad \mu,\nu = 0,1,2,3$$

where $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric tensor of Special Relativity.

(10.3) Contraction of Dirac γ-matrices *(3 points)*

Show (without using a particular representation of the Dirac γ-matrices but only their anticommutator relation) that

(i) $\gamma^\mu\gamma_\mu = 4$,

(ii) $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4g^{\alpha\beta}$,

(iii) $\gamma_\mu \not{A}\gamma^\mu = -2\not{A}$,

where $\not{A} = \gamma^\mu A_\mu = \gamma_\mu A^\mu$ ("Feynman dagger" or "Feynman slash notation") with A^μ an arbitrary four-vector, and $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric tensor.

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(10.4) Repetition: Tensor calculus

Consider a coordinate transformation $x \to \bar{x} = f(x)$ and the inverse $\bar{x} \to x = h(\bar{x})$ (here, we collect all $x^1, x^2, \ldots$ in x, and all $\bar{x}^1, \bar{x}^2, \ldots$ in $\bar{x}$).

Per definition, a *scalar field* transforms according $\bar{\varphi}(\bar{x}) = \varphi(x)$, i.e., for the same point represented by x in one coordinate system and $\bar{x}$ in the other, $\varphi(x)$ and $\bar{\varphi}(\bar{x})$ have the same numerical value.

The differentials $\mathrm{d}\bar{x}^\mu$ transform according $\mathrm{d}\bar{x}^\mu = \frac{\partial f^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu$ (summation over ν), which we write as

$$\mathrm{d}\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu. \tag{1}$$

By definition, any n-tuple A^μ which transforms like (1),

$$\bar{A}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}\,A^\nu, \tag{2}$$

is a *contravariant vector*.

Instead, the partial derivatives of a scalar field transform according

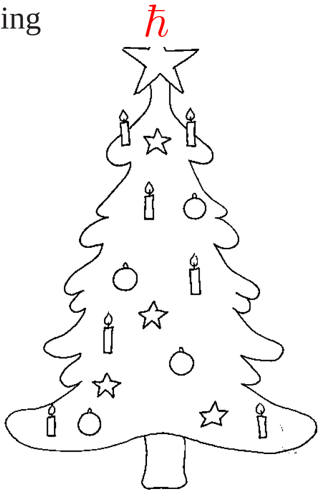
$$\frac{\partial \bar{\varphi}(\bar{x})}{\partial \bar{x}^\mu} = \frac{\partial \varphi(x)}{\partial \bar{x}^\mu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu}\frac{\partial \varphi(x)}{\partial x^\nu}. \tag{3}$$

By definition, any n-tuple B_μ which transforms like (3),

$$\bar{B}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu}\,B_\nu, \tag{4}$$

is a *covariant vector*.

Show that $\bar{S} = \bar{A}^\mu \bar{B}_\mu = A^\mu B_\mu = S$, i.e., $A^\mu B_\mu$ is a scalar.



(10.5) ***Christmas quiz** How are the four photos related?