Advanced Theoretical Physics
WS 09/10

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Seminar 4

Keywords:

Adjoint, hermitian, unitary, and projection operators,
eigenvalues, eigenvectors, degeneracy

Questions:

1. Let us assume you calculated the eigenvalues and eigenvectors of an operator $\hat{L}$. What are the eigenvalues and eigenvectors of $\hat{L}^{-1}$ and $\hat{L}^{\dagger}$?
2. A colleague tells you that an eigenvalue of an unitary operator is 42. Is this possible?
3. Given a discrete set of non-orthogonal and non-normalized basis vectors, how can you generate an orthonormalized basis set out of it?

Assignment 4

(due November 9, 2009)

Operators

4.1 Prove the following identity and statements for an arbitrary operator $\hat{L}$, hermitian operators $\hat{H}$ and $\hat{K}$, and unitary operator $\hat{U}$:

$$(\hat{L}^{-1})^{\dagger} = (\hat{L}^{\dagger})^{-1}, \quad \hat{H}\hat{K}+\hat{K}\hat{H} \text{ is hermitian}, \quad \mathrm{e}^{\mathrm{i}\hat{H}} \text{ is unitary}, \quad \mathrm{i}\,\frac{\hat{U}-1}{\hat{U}+1} \text{ is hermitian.}$$

4.2 Show that the matrix

$$\frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

(with α, β complex numbers) is unitary.

Projectors

4.3 Let $\hat{P}_{\Omega} = \int_{\Omega} |\alpha\rangle\langle\alpha|\,\mathrm{d}\alpha$ be the projector on a subspace Ω. Show that $\hat{P}_{\Omega}$ is independent of the particular choice of the basis vectors $|\alpha\rangle$ spanning Ω, and that $\hat{P}_{\Omega}$ is both self-adjoint and idempotent.

An application

4.4 What is the adjoint of the operator $|u\rangle\langle v|$? Determine $(1+|u\rangle\langle v|)^{-1}$ and use it to solve the integral equation

$$\varphi(x) + u(x)\int_a^b v^*(x')\varphi(x')\,\mathrm{d}x' = \chi(x)$$

for $\varphi(x)$.