Lecture Notes

Plasma & Astrophysics

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Literature

Fundamental plasma physics is a broad but mature field. As a consequence, the textbook-level literature differs mostly in the style of presentation and the selection of and the emphasis on the various topics. I shall necessarily follow the same path. After unfolding the fundamentals, I will stress the subjects I find most important and interesting, choosing the way of presentation I consider most didactic and elegant. The following list contains textbooks I use myself for the preparation of my lectures. It will be updated occasionally. The absence of a book from this list doesn't mean I don't like it, just that I don't have it on my shelve.

General

- Francis F. Chen, *Plasma Physics and Controlled Fusion*
- Setsuo Ichimaru, *Basic Principles of Plasma Physics*
- Nicholas A. Krall and Alvin W. Trivelpiece, *Principles of Plasma Physics*
- Russel M. Kulsrud, *Plasma Physics for Astrophysics*
- Boris M. Smirnov, *Fundamentals of Ionized Gases*
- Karl-Heinz Spatschek, *Theoretische Plasmaphysik*

Specialized on Certain Aspects of Plasma Physics

- Matthias Bartelmann, *Theoretical Astrophysics*
- C.K. Birdsall and A.B. Langdon, *Plasma Physics via Computer Simulation*
- Shalom Eliezer, *The Interaction of High-Power Lasers with Plasmas*
- Paul Gibbon, *Short Pulse Laser Interactions with Matter*
- William L. Kruer, *The Physics of Laser Plasma Interactions*
- Peter Mulser and Dieter Bauer, *High Power Laser-Matter Interaction*
- Thomas H. Stix, *The Theory of Plasma Waves*

CONTENTS

INTRODUCTION

- *Plasma* is a quasi-neutral gas of charged particles. In addition, there may be neutral particles present.
- Let us consider the case when there are just electrons (mass m and charge $-e < 0$) and ions of one type (mass M and charge $Q > 0$) interacting with each other and with an external field described by the vector potential $\mathbf{A}$ and a scalar potential Φ. The non-relativistic Hamiltonian for such a system,[1]

$$\begin{aligned} H &= \sum_I \left\{ \frac{[\mathbf{P}_I - Q\mathbf{A}(\mathbf{R}_I, t)]^2}{2M} + Q\Phi(\mathbf{R}_I, t) \right\} \\ &\quad + \sum_i \left\{ \frac{[\mathbf{p}_i + e\mathbf{A}(\mathbf{r}_i, t)]^2}{2m} - e\Phi(\mathbf{r}_i, t) \right\} \\ &\quad + \frac{1}{2} \sum_{I \neq J} \frac{Q^2}{4\pi\epsilon_0 |\mathbf{R}_I - \mathbf{R}_J|} + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_j|} \\ &\quad - \sum_{iI} \frac{Qe}{4\pi\epsilon_0 |\mathbf{R}_I - \mathbf{r}_i|}, \end{aligned} \tag{1}$$

  (upper case indices, positions and momenta refer to ions, lower case to electrons) does not look different from the Hamiltonian for a molecule or an ionic solid (in external fields).[2] Hence the emphasis on *quasi-neutral gas* above.
- The notion of *gas* implies that there should be *free* ions and electrons. If they were bound to form a neutral gas of atoms we would call it an ordinary gas. We have studied ordinary, ideal gases (and also non-ideal ones that may exhibit phase transitions) in Thermodynamics and Statistical Physics.
- Besides gas, liquid, and solid, plasma is sometimes called the *fourth state of matter*. In fact, among the visible matter in the universe, plasma is the most common state of matter.

[1] No operator hats in this lecture!

[2] Any non-relativistic matter made of atomic nuclei and electrons (in external fields) is described by (1), as long as it is permitted to describe the effect of the nuclei effectively by a Coulomb potential for the nuclear charge $Q = Ze$.

- Free electrons and ions in a gas are generated by *ionization*. In order to remove an electron from an initially neutral atom, an electric field $\mathbf{E}$ is required that gives rise to a force $-e\mathbf{E}$. Such a field can be provided externally (e.g., by charging up a capacitor until a *gas discharge* occurs) or internally, by close encounters of sufficiently energetic particles (i.e., collisions).

- If particles collide with an energy greater than the ionization potential $|\epsilon_\mathrm{i}|$ electrons e^- could be emitted in inelastic processes like

$$A + B \longrightarrow A^+ + B + e^- \qquad (2)$$
$$C + e^- \longrightarrow C^+ + 2e^- . \qquad (3)$$

- In the Statistical Physics lecture we derived the Saha equation, which tells us the ratio of charged to neutral particles,

$$\frac{n_\mathrm{i}}{n_\mathrm{n}} \simeq 2.4 \times 10^{21} \frac{T^{3/2}}{n_\mathrm{i}} \, \mathrm{e}^{-|\epsilon_\mathrm{i}|/k_\mathrm{B}T} \qquad (4)$$

($n_\mathrm{i,n}$: ion and neutral particle density in m^{-3}; T: temperature in K; k: Boltzmann constant). The fractional ionization degree is $n_\mathrm{i}/(n_\mathrm{n}+n_\mathrm{i})$. As $|\epsilon_\mathrm{i}|$ is typically a couple of eV, the fractional ionization degree in our human environment (i.e., at the surface of the earth where typically $-60 < T < 40\,^\circ\mathrm{C}$) is negligibly low. If it weren't we wouldn't be here.

[E] Calculate the fractional ionization degree of air at $T = 300\,\mathrm{K}$. Consider nitrogen ($|\epsilon_\mathrm{i}| = 14.5\,\mathrm{eV}$) and use $n_\mathrm{n} = 3 \times 10^{25}\,\mathrm{m}^{-3}$.

- The fact that plasma is made of charged particles which interact via the *long-range* Coulomb force has drastic consequences. It leads to rich dynamics, especially *collective behavior*.

- As in a metal[3] charge imbalances create electric fields to which the plasma quickly[4] responds in order to remove them. As a consequence, the plasma stays *quasi-neutral*.

- Plasmas occur on all length scales in physics. Cosmological, galactic, stellar, inside planets, in supernovae, in the early universe. On earth,

[3] The electrons and the positive charge background in a metal may, in fact, be considered a plasma.

[4] What "quickly" means will become clear later on.

in thunderbolts, neon lamps, flames, ionosphere, polar light, nuclear bombs, nuclear fusion reactors, and whenever intense laser pulses hit matter. Microscopically, quantum plasmas in metals, semiconductors (e^--hole plasmas, excitons), electron-positron, nuclear matter and quark-gluon plasmas (the latter two not purely coulombic).

E Plasmas are often characterized according to where they "reside" in a log-density-log-temperature plot. Make one and indicate the regions of (a) flames, (b) sparks (e.g., in a starter), (c) solar photosphere, (d) solar corona, (e) earth's ionosphere E-layer, (f) lightning, (g) solar wind near the earth, (h) magnetic confinement fusion plasmas in tokamaks, (i) laser inertial confinement fusion plasmas, (j) metals. You may "google" for the respective numbers.

1

BASIC PROPERTIES OF PLASMAS

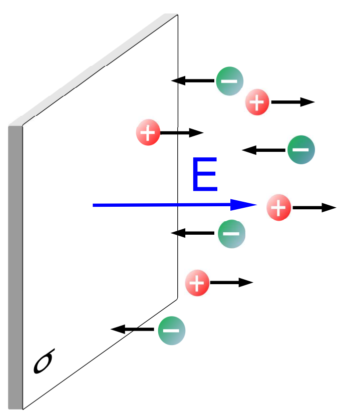
- We will now investigate the relevant length and time scales of plasmas. In the context of the above mentioned *quasi-neutrality* the question arises over which length scales plasmas are able to *screen* immersed test charges in order to maintain this quasi-neutrality. Moreover, if a charge is immersed suddenly, how long does it take to screen? In such a situation one could actually expects an "overshooting" of the screening cloud, leading to *plasma oscillations*. Wiggling a test charge in a plasma then should lead to *plasma waves*.

1.1 DEBYE LENGTH

- Consider a thin plate with surface charge density $\sigma > 0$ immersed into an initially homogeneous plasma of ion number[1] density $n_{i,0}$ and electron number density $n_{e,0}$,

$$Zn_{i,0} = n_{e,0} \tag{5}$$

(Z: ion charge state). We assume that the temperatures are equilibrated, $T = T_i = T_e$.



[1] We omit "number" in the following. n typically stands for number densities, ρ for charge densities.

- We further assume that the plate is covered with an insulating layer to prevent electrons or ions from actually hitting the plate (i.e., leading to an electrical short).
- For an (in the two lateral directions) infinitely extended plate the electric field in vacuum follows from GAUSS' law and would be just constant: $E = \sigma/2\epsilon_0$.
- When the plate is immersed in the plasma, screening sets in, leading to a spatially varying field $E(x)$ that needs to be calculated self-consistently. Introducing the potential $\Phi(x)$,

$$E(x) = -\frac{\partial}{\partial x}\Phi(x), \tag{6}$$

  and assuming BOLTZMANN distributions

$$n_{\mathrm{e}}(x) = n_{\mathrm{e},0}\ \mathrm{e}^{e\Phi(x)/k_{\mathrm{B}}T}, \qquad \lim_{x\to\infty} n_{\mathrm{e}}(x) = n_{\mathrm{e},0},\ \lim_{x\to\infty}\Phi(x) = 0, \tag{7}$$

$$n_{\mathrm{i}}(x) = n_{\mathrm{i},0}\ \mathrm{e}^{-Ze\Phi(x)/k_{\mathrm{B}}T}, \qquad \lim_{x\to\infty} n_{\mathrm{i}}(x) = n_{\mathrm{i},0}, \tag{8}$$

  the POISSON equation

$$\epsilon_0\nabla^2\Phi(x) = -\rho(x) \tag{9}$$

  gives

$$\epsilon_0\frac{\partial^2}{\partial x^2}\Phi(x) = -[Zen_{\mathrm{i}}(x) - en_{\mathrm{e}}(x)] = en_{\mathrm{e},0}\left[\mathrm{e}^{e\Phi(x)/k_{\mathrm{B}}T} - \mathrm{e}^{-Ze\Phi(x)/k_{\mathrm{B}}T}\right], \tag{10}$$

  where we have used (5).
- Equation (10) is a nonlinear differential equation for the boundary value problem of determining $\Phi(x)$. If $\forall x$

$$Ze\Phi(x) \ll k_{\mathrm{B}}T \tag{11}$$

  we can proceed analytically and find

$$\frac{\partial^2}{\partial x^2}\Phi(x) = \frac{e^2 n_{\mathrm{e},0}(Z+1)}{\epsilon_0 k_{\mathrm{B}}T}\,\Phi(x) \tag{12}$$

  so that

$$\Phi(x) = \Phi_0\ \mathrm{e}^{-x/\lambda_{\mathrm{D}}} \tag{13}$$

  with

$$\lambda_{\mathrm{D}} = \left(\frac{\epsilon_0 k_{\mathrm{B}}T}{e^2 n_{\mathrm{e},0}(Z+1)}\right)^{1/2}. \tag{14}$$

E Φ_0 is the potential (i.e., voltage) at $x = 0$, i.e. directly at the plate. This voltage may be kept constant by connecting the plate to a battery. But how do you determine Φ_0 if an isolated plate with fixed σ is immersed?

- If we are interested in screening processes so fast that the ions are too inert to redistribute (10) becomes

$$\epsilon_0 \frac{\partial^2}{\partial x^2}\Phi(x) = -[Zen_{\mathrm{i},0} - en_{\mathrm{e}}(x)] = en_{\mathrm{e},0}\left[\mathrm{e}^{e\Phi(x)/k_\mathrm{B}T} - 1\right], \tag{15}$$

leading also to (13) but with

$$\boxed{\lambda_\mathrm{D} = \left(\frac{\epsilon_0 k_\mathrm{B} T}{e^2 n_{\mathrm{e},0}}\right)^{1/2}}. \tag{16}$$

- (16) is called Debye *length*.[2] Keep in mind the assumptions that led to (16).

- The Debye length tells us over which distance the potential drops by $1/\mathrm{e}$. The bigger the electron density, the more electrons can participate in the screening process and thus the smaller λ_D. The bigger the temperature, the higher the pressure which counteracts the accumulation of electrons around the test charge, and thus the bigger λ_D.

E Sketch qualitatively $\Phi(x)$, $n_\mathrm{e}(x)$, and $n_\mathrm{i}(x)$ (the latter for (i) mobile ions and (ii) immobile ions).

- The same result for λ_D is found in spherical geometry. The potential due to an, e.g., immersed charge q is

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r}\,\mathrm{e}^{-r/\lambda_\mathrm{D}}. \tag{17}$$

Here it becomes obvious that condition (11) (here for $Z = 1$),

$$e\Phi(r) \ll k_\mathrm{B}T \tag{18}$$

cannot be fulfilled for point charges as $r \to 0$. We thus expect (17) to be valid for $r > r_0$ where r_0 is chosen such that $e\Phi(r_0) \ll k_\mathrm{B}T$, and q is a net charge inside the sphere of radius r_0.

- The potential (17) is called Debye *potential*.

[2] Or Debye-Hückel length.

1.2 PLASMA PARAMETER

- The number of particles N_{D} actually contributing to screening is called *plasma parameter*. It is of the order of

$$N_{\mathrm{D}} = n\,\frac{4}{3}\pi\lambda_{\mathrm{D}}^{3} \sim T^{3/2}n^{-1/2}, \tag{19}$$

  where the density n is n_{e}, n_{i} or any other particle type of interest. Note that (for fixed temperature) this number decreases with increasing density.

E Calculate the proportionality factor in (19) if T is given in K and n in m^{-3}.

- We expect a *continuum description* of a plasma (i.e., as charge *distributions* or *fluids*) to be valid if very many particles contribute to screening,

$$N_{\mathrm{D}} \gg 1. \tag{20}$$

- Condition (20) follows automatically if we require the order of the mean kinetic energy $k_{\mathrm{B}}T$ to be much greater than the order of the mean potential energy $e^2/4\pi\epsilon_0 r_{\mathrm{WS}}$ in the plasma. Here we take as the typical distance between particles the WIGNER-SEITZ radius

$$n\,\frac{4}{3}\pi r_{\mathrm{WS}}^{3} = 1 \quad \Rightarrow \quad r_{\mathrm{WS}} = \left(\frac{3}{4n\pi}\right)^{1/3}. \tag{21}$$

  Hence

$$1 \ll \Gamma^{-1} = \frac{k_{\mathrm{B}}T}{e^2}\,4\pi\epsilon_0\left(\frac{3}{4n\pi}\right)^{1/3} \sim Tn^{-1/3} \quad \Rightarrow \quad N_{\mathrm{D}} \ggg 1. \tag{22}$$

- Γ is called the *coupling parameter*. If $N_{\mathrm{D}} \gg 1$, i.e., $\Gamma \ll 1$, the plasma is *"ideal"*, i.e., it is *weakly coupled*.[3]

E Calculate λ_{D}, N_{D}, and Γ for some of the plasmas in your log-density-log-temperature plot. Which of them are ideal?

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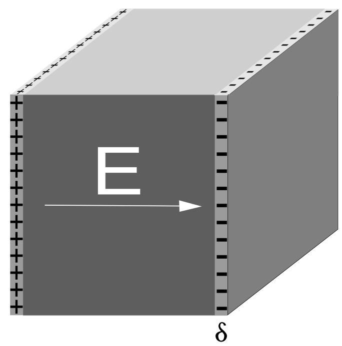
[3] The definitions of the plasma parameter and the coupling parameter may differ by numerical factors from textbook to textbook.

1.3 PLASMA FREQUENCY

- Consider a plasma slab of constant electron and ion density. We are interested with which frequency the electrons oscillate if the system is perturbed. Consider, for instance, a shift of the electron slab by δ:



- The resulting electric field is the same as that of a plane capacitor, $E = en_{e,0}\delta/\epsilon_0$ so that the restoring force on the electron cloud is $F = -eE = -e^2 n_{e,0}\delta/\epsilon_0$, i.e., linear in the excursion δ. The result will thus be harmonic oscillations of frequency ω_p,

$$F = -m_e \omega_p^2 \, \delta = -\frac{e^2 n_{e,0} \delta}{\epsilon_0} \tag{23}$$

so that

$$\boxed{\omega_p = \left(\frac{e^2 n_{e,0}}{\epsilon_0 m_e} \right)^{1/2}} \,. \tag{24}$$

- The frequency (24) is called *electron plasma frequency*. It is of fundamental importance and sets the time scale on which transient charge imbalances are screened.

E Calculate ω_p for some of the plasmas in your log-density-log-temperature plot.

E Which oscillation frequency do you get if you compress the electron slab by δ instead of displacing it rigidly?

E Which oscillation frequency do you get for a spherically symmetric setup?

2

PARTICLE MOTION IN EXTERNAL FIELDS

- Plasmas consist of charged particles that move in an electromagnetic field. Although "there is but one field, and Maxwell is its prophet" it often makes sense to decompose the field into the external, "applied" one and the one that is generated by the plasma particles responding to this external field (and to each other). Examples for external electromagnetic fields acting on plasmas are the confining fields in a magnetic fusion reactor, the earth's magnetic field, the fields in plasma propulsion systems for space travel, or intense laser pulses interacting with matter. The external field—from the view-point of one plasma—could be generated by another plasma. In astrophysics this is actually a common situation (think about the earth's magnetic field).

- A first step towards the understanding of the extremely rich, complex, and intriguing plasma dynamics is the study of how *individual* charges move in a *given* external field.

- The Lorentz force on a particle of charge q in the form

$$\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{25}$$

  is even relativistically correct. However, in this chapter we will consider mostly non-relativistic motion where

$$m\dot{\mathbf{v}} = m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{26}$$

2.1 SPATIALLY AND TEMPORALLY CONSTANT FIELDS

- If $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} = B\mathbf{e}_z$, $B \geq 0$, we find the equations of motion

$$\dot{v}_x = \frac{qB}{m} v_y, \qquad \dot{v}_y = -\frac{qB}{m} v_x, \qquad \dot{v}_z = 0, \tag{27}$$

$$\Rightarrow \qquad \ddot{v}_{x,y} = -\left(\frac{qB}{m}\right)^2 v_{x,y}, \qquad v_z = v_\parallel = \text{const} \in \mathbb{R}$$

so that[1]

$$v_{x,y} = v_\perp \, \mathrm{e}^{\pm \mathrm{i}\omega_\mathrm{c} t}, \qquad v_\perp = \text{const} \in \mathbb{C} \tag{28}$$

with the *cyclotron frequency*

$$\omega_\mathrm{c} = \frac{|q|B}{m} \geq 0. \tag{29}$$

If we choose $v_\perp$ such that

$$v_x = v_\perp \, \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t} \quad \Rightarrow \dot{v}_y = -\frac{qB}{m} v_\perp \, \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t}, \tag{30}$$

and the position coordinates are

$$x = x_0 - \mathrm{i}\frac{v_\perp}{\omega_\mathrm{c}} \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t}, \qquad y = y_0 + \frac{qB}{m} \frac{v_\perp}{\omega_\mathrm{c}^2} \, \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t} = y_0 + \frac{q}{|q|} \frac{v_\perp}{\omega_\mathrm{c}} \, \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t}. \tag{31}$$

Writing $v_\perp = |v_\perp| \, \mathrm{e}^{\mathrm{i}\varphi_{v_\perp}}$ we finally obtain

$$x = x_0 + r_\mathrm{L} \, \sin(\omega_\mathrm{c} t + \varphi_{v_\perp}), \qquad y = y_0 + \frac{q}{|q|} r_\mathrm{L} \cos(\omega_\mathrm{c} t + \varphi_{v_\perp}) \tag{32}$$

with the LARMOR radius

$$r_\mathrm{L} = \frac{|v_\perp|}{\omega_\mathrm{c}} = \frac{|p_\perp|}{|q|B}, \tag{33}$$

where $p_\perp$ is the momentum $\perp \mathbf{B}$.

- We find circular orbits of radius r_L, i.e., the particles *gyrate* about the magnetic field lines. The heavier ions will have a smaller cyclotron frequency but a bigger LARMOR radius than the lighter electrons (for a given $v_\perp$ for both). Moreover, the direction of rotation is opposite for ions and electrons ($q/|q| = \mathrm{sgn}(q)$ in the equation for y). They orbit such that the magnetic field they generate opposes the external magnetic field.

- The point (x_0, y_0, z) with $z = z_0 + v_\parallel t$ is called the *guiding center*. If one "averages-out" the cyclotron motion, only the *drift* of the guiding center remains.

[1] Physical quantities are obtained by taking the real part.

- Relativistically, we have

$$\dot{\mathbf{p}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\gamma\mathbf{v}) = q\mathbf{v}\times\mathbf{B}, \qquad \gamma = (1-\beta^2)^{-1/2}, \quad \beta = v/c \tag{34}$$

with c the speed of light. "Dotting" with $\mathbf{v}$ yields

$$m\mathbf{v}\cdot(\dot{\gamma}\mathbf{v}+\gamma\dot{\mathbf{v}}) = 0.$$

Writing $\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}v^2}\frac{\mathrm{d}v^2}{\mathrm{d}t}$ this can be written as

$$m\left(v^2\frac{\mathrm{d}\gamma}{\mathrm{d}v^2}+\frac{1}{2}\gamma\right)\frac{\mathrm{d}v^2}{\mathrm{d}t} = 0.$$

The expression in the brackets is positive. Hence

$$v^2 = \text{const} \quad \Rightarrow \quad \gamma = \text{const} \quad \Rightarrow \quad \Rightarrow \quad E = \gamma mc^2 = \text{const}.$$

The last expression is just the total relativistic energy of the particle. One may introduce the effective relativistic mass

$$\tilde{m} = \gamma m = \text{const}$$

so that the relativistic equation of motion (34) formally looks like the non-relativistic one,

$$\tilde{m}\dot{\mathbf{v}} = q\mathbf{v}\times\mathbf{B},$$

and we can use the non-relativistic results derived above.

- This only works without an electric field present!

- At relativistic velocities the cyclotron frequency decreases according

$$\omega_{\mathrm{c}} = \frac{|q|B}{\gamma m}. \tag{35}$$

- The Larmor radius increases:

$$r_{\mathrm{L}} = \frac{|v_\perp|}{\omega_{\mathrm{c}}} = \frac{|v_\perp|\gamma m}{|q|B} = \frac{|p_\perp|}{|q|B}. \tag{36}$$

Because γ (and thus $p_\perp$) also depends on $v_\parallel$ the relativistic Larmor radius depends on it as well. One can write

$$r_{\mathrm{L}} = \frac{|p\sin\theta|}{|q|B} \tag{37}$$

where $p = |\mathbf{p}|$ is the total relativistic momentum and θ is the angle between $\mathbf{p}$ and $\mathbf{B}$.

2.1.1 *E×B drift*

- Let us now allow for a non-vanishing but constant **E**. We can rotate the coordinate system such that $E_y = 0$. The equations of motion (27) become

$$\dot{v}_x = \pm\omega_c\, v_y + \frac{q}{m}E_x, \qquad \dot{v}_y = \mp\omega_c\, v_x, \qquad \dot{v}_z = \frac{q}{m}E_z \tag{38}$$

  where the upper sign is for $q/|q| = 1$, the lower for $q/|q| = -1$.

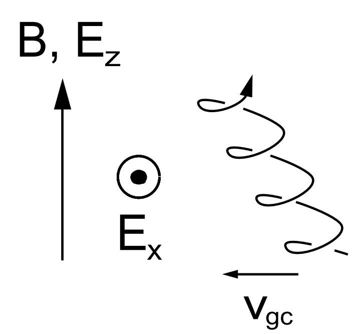
E Show that similarly as above for the case without electric field we obtain from (38)

$$v_x = v_\perp\, e^{i\omega_c t}, \qquad v_y = \pm i v_\perp\, e^{i\omega_c t} + v_{gc}, \qquad v_z = \frac{q}{m}E_z t + v_\parallel. \tag{39}$$

- The velocity

$$v_{gc} = -\frac{E_x}{B} \tag{40}$$

  is the new feature here, namely a drift velocity of the guiding center perpendicular to both **B** and **E** due to the component of the electric field that is perpendicular to **B**.



- In a coordinate-free manner we can write this as

$$\mathbf{v}_{gc}^{\perp} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}. \tag{41}$$

- Note that this drift only depends on the external fields but neither on the particle properties q and m nor on the initial velocity (determining $v_\perp$). Electrons and ions may drift in the same direction.

- Instead of $q\mathbf{E}$ there may be other forces $\mathbf{F}$ acting on the particle, e.g., gravity, gradient forces or centrifugal forces because of curved field lines (see next section below). The above calculation remains essentially the same but now a dependence on the charge arises,

$$\boxed{\mathbf{v}_{\mathrm{gc}} = \frac{\mathbf{F}\times\mathbf{B}}{qB^2}}, \tag{42}$$

so that electrons and ions will drift in opposite directions.

2.2 GRAD-B DRIFT

- Let us first consider the case where still $\mathbf{B} = B_z\mathbf{e}_z$ but $B_z = B = B(y)$, i.e., the field strength varies as a function of the position. We expand

$$B(y) = B_0 + \underbrace{\left.\frac{\partial B}{\partial y}\right|_{y=0}}_{B'} y + \cdots \tag{43}$$

and assume that the scale length $L = (|\partial B/\partial y|/|B|)^{-1}$ over which B changes is much larger than r_{L}.

- Equations (27) become

$$\dot{v}_x = \frac{q(B_0 + B'y)}{m} v_y, \qquad \dot{v}_y = -\frac{q(B_0 + B'y)}{m} v_x, \qquad \dot{v}_z = 0. \tag{44}$$

- In order to figure out the drift induced by B' we time-average (indicated by $\langle\cdot\rangle$) the forces over the gyration period. Further, we plug-in the solutions for $B' = 0$ from above (where we choose, without loss of generality, $\varphi_{v_\perp} = x_0 = y_0 = 0$):

$$\langle\dot{v}_x\rangle = 0, \quad \langle\dot{v}_y\rangle = -\left\langle \frac{qB'\frac{q}{|q|}r_{\mathrm{L}}\cos\omega_{\mathrm{c}}t}{m} v_\perp \cos\omega_{\mathrm{c}}t \right\rangle = \mp\frac{qB'r_{\mathrm{L}}v_\perp}{2m}, \quad \langle\dot{v}_z\rangle = 0$$

(nothing for $\langle\dot{v}_x\rangle$ because $\langle\sin\cdot\cos\cdot\rangle = 0$) so that the only non-vanishing force component is

$$\langle F_y\rangle = \mp\frac{1}{2}qB'r_{\mathrm{L}}v_\perp, \tag{45}$$

i.e., both positive and negative charges are repelled by an increasing B.

- From (42) we know already that such an additional force gives rise to a drift
$$\mathbf{v}_{\mathrm{gc,grad}} = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} = \mp \frac{B' r_{\mathrm{L}} v_\perp}{2B} \mathbf{e}_x. \tag{46}$$

- For ∇B in any direction we obtain the general expression for the so-called *grad-B drift*
$$\mathbf{v}_{\mathrm{gc},\nabla B} = \pm \frac{1}{2} r_{\mathrm{L}} v_\perp \frac{\mathbf{B} \times \nabla B}{B^2}. \tag{47}$$

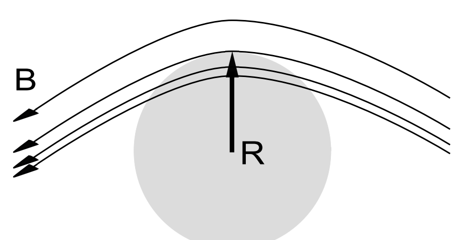
- Note the physical origin of that drift: while gyrating, the particles "sense" the gradient of the magnetic field because they spend less time in the region of higher B (on an orbit with smaller radius) than in the region of lower B.

- The direction of the grad-B drift is opposite for electrons and ions.

2.3 CURVATURE DRIFT

- Let us now consider the case where $\mathbf{F}$ in (42) is the centrifugal force
$$\mathbf{F}_{\mathrm{cf}} = \frac{m v_\parallel^2}{R} \mathbf{e}_r = m v_\parallel^2 \frac{\mathbf{R}}{R^2}. \tag{48}$$

  Here, we assume curved magnetic field lines with a curvature radius along which in zeroth order the particles drift along with $v_\parallel$ and undergo their gyro-motion with $v_\perp$.



- Equation (42) gives the curvature drift
$$\mathbf{v}_{\mathrm{gc,curv}} = \frac{m v_\parallel^2}{qB^2} \frac{\mathbf{R} \times \mathbf{B}}{R^2}. \tag{49}$$

- However, a curved magnetic field cannot be constant in magnitude because $\nabla \times \mathbf{B} = \mathbf{0}$ has to be fulfilled in vacuum (and without a time-dependent electric field present). In fact, we know that an azimuthal magnetic field around a current in z-direction decreases as $\sim r^{-1}$ with increasing r. As a consequence the curvature drift is necessarily accompanied by a grad-B drift. As $\nabla B/B = -\mathbf{R}/R^2$ we find

$$\mathbf{v}_{\mathrm{gs},\nabla B} = \mp\frac{1}{2}\frac{v_\perp r_\mathrm{L}}{B^2}\mathbf{B}\times B\frac{\mathbf{R}}{R^2} = \pm\frac{1}{2}\frac{v_\perp^2}{\omega_\mathrm{c}}\frac{\mathbf{R}\times\mathbf{B}}{R^2 B} = \frac{1}{2}\frac{mv_\perp^2}{q}\frac{\mathbf{R}\times\mathbf{B}}{R^2B^2}$$

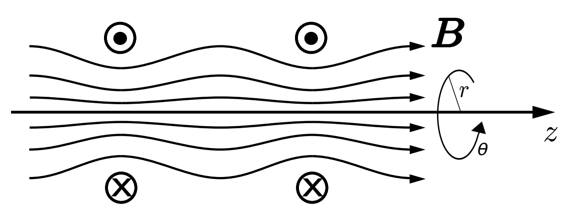
so that the total drift reads

$$\mathbf{v}_{\mathrm{gc,curv,grad}} = \frac{m}{q}\frac{\mathbf{R}\times\mathbf{B}}{R^2B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right). \tag{50}$$

- Imagine you want to trap charged particles by a magnetic field. You may want to create a torus of magnetic field lines (as in a tokamak fusion reactor). The drift (50) is bad news for you: electrons and ions will drift out perpendicular to **R** and **B**, in opposite directions.

2.4 MAGNETIC MIRROR

- We will now discuss a set-up by which charged particles can be better magnetically confined.
- Consider the magnetic field lines of a pair of coils.



- B_z clearly varies as a function of z. We assume $\partial B_z/\partial z$ is given at $r = 0$ and does not change much as a function of r.
- Because of $\nabla\cdot\mathbf{B} = (1/r)\partial(rB_r)/\partial r + \partial B_z/\partial z = 0$, the B_r-component follows,

$$B_r = -\frac{1}{2}r\frac{\partial B}{\partial z}. \tag{51}$$

E Show that the grad-B drift will not lead to a loss of particles in this set-up.

- The r and θ components of the LORENTZ force lead to the usual local gyro-motion around the field lines. But there is also a force in z-direction:

$$F_z = -qv_\theta B_r = \frac{1}{2} q v_\theta r \frac{\partial B}{\partial z}. \tag{52}$$

- On the z-axis we have $v_\theta = \mp v_\perp$ for $q = \pm|q|$ and $r = r_\mathrm{L}$ so that

$$F_z = \mp \frac{1}{2} q v_\perp r_\mathrm{L} \frac{\partial B}{\partial z} = -\frac{1}{2} \frac{m v_\perp^2}{B} \frac{\partial B}{\partial z}. \tag{53}$$

- With the *magnetic moment*

$$\boxed{\mu = \frac{1}{2} \frac{m v_\perp^2}{B}} \tag{54}$$

this can be written in the familiar way

$$F_z = -\mu \frac{\partial B}{\partial z}. \tag{55}$$

E Show that (54) is consistent with the usual definition of the magnetic moment $\mu = IA$ of a current loop of area A and current I.

- The result (55) can be generalized to motion along off-axis field lines,

$$F_\| = -\mu \partial_\| B. \tag{56}$$

- With the coordinate along the field line s we have $v_\| = \mathrm{d}s/\mathrm{d}t$ and

$$m \frac{\mathrm{d}v_\|}{\mathrm{d}t} = -\mu \frac{\partial B}{\partial s} \qquad \Rightarrow \qquad m v_\| \frac{\mathrm{d}v_\|}{\mathrm{d}t} = -\mu \frac{\partial B}{\partial s} \frac{\mathrm{d}s}{\mathrm{d}t}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\|^2 \right) = -\mu \frac{\mathrm{d}B}{\mathrm{d}t} \tag{57}$$

where $\mathrm{d}B/\mathrm{d}t$ is the change in B the particle experiences as it moves along the field line (note that at a fixed point in space $\mathrm{d}B/\mathrm{d}t = 0$).

- Energy conservation requires that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\|^2 + \frac{1}{2} m v_\perp^2 \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\|^2 + \mu B \right) = 0. \tag{58}$$

As a result,

$$\frac{d\mu}{dt} = 0 \tag{59}$$

follows.

The magnetic moment is conserved. If the particle moves into regions of stronger B it gyrates faster (i.e., $v_\perp$ increases). As a consequence, $v_\parallel$ has to decrease. In other words, the force (56) repels charged particles from regions of increasing field strength, independent of the sign of the charge. In that way particles can be trapped between two magnetic mirrors, as in the two-coil set-up mentioned in the beginning. Such a setup is called a *magnetic bottle*.

- As for $\mu = \text{const}$ follows $v_\perp \sim \sqrt{B}$ we have because of $\omega_c \sim B$ that $r_L \sim 1/\sqrt{B}$, i.e., the LARMOR radius decreases in regions of higher B. In fact, the magnetic flux through the area enclosed by a LARMOR orbit is $\sim Br_L^2$ and thus constant (see below).

E Show that particles with too small $v_\perp/v_\parallel$-ratio will not be trapped (*loss cone*).

2.5 ADIABATIC COMPRESSION

- Let us consider a spatially uniform but time-dependent magnetic field $\mathbf{B}(t)$ which by

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{60}$$

is accompanied by an electric field.

- Electric fields can do work on the plasma, magnetic fields cannot. Multiplying the LORENTZ force $m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ by $\mathbf{v}_\perp$ gives

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\mathbf{v}_\perp \cdot (\mathbf{E} + \underbrace{\mathbf{v}_\perp \times \mathbf{B}}_{\perp \mathbf{v}_\perp}) = q\mathbf{E} \cdot \mathbf{v}_\perp. \tag{61}$$

- Let $\mathbf{l}$ be the gyration path,

$$\frac{d\mathbf{l}}{dt} = \mathbf{v}_\perp. \tag{62}$$

The change in the kinetic gyro-motion energy over one gyro period then is

$$\Delta\left(\frac{1}{2}mv_\perp^2\right) = \int_0^{2\pi/\omega_c} q\mathbf{E} \cdot \frac{d\mathbf{l}}{dt}\, dt. \tag{63}$$

- If **B** changes slowly in time over one such period we can approximate **l** by the unperturbed, circular **l**,

$$\Delta\left(\frac{1}{2}mv_\perp^2\right) = \oint q\mathbf{E}\cdot\mathrm{d}\mathbf{l} = q\int(\nabla\times\mathbf{E})\cdot\mathrm{d}\mathbf{n} = -q\frac{\mathrm{d}}{\mathrm{d}t}\int\mathbf{B}\cdot\mathrm{d}\mathbf{n}. \qquad (64)$$

Here, d**n** is the oriented surface normal vector, and the integration over the area enclosed by the circular orbit **l** just yields πr_L^2. For positive charges, d**n** is anti-parallel to **B**, for negative charges parallel. Hence

$$\Delta\left(\frac{1}{2}mv_\perp^2\right) = \pm q\dot{B}\pi r_\mathrm{L}^2 = \mu\,\Delta B \qquad (65)$$

where μ is the magnetic moment (54) and

$$\Delta B = \frac{2\pi\dot{B}}{\omega_\mathrm{c}} \qquad (66)$$

is the change in the magnetic field over one gyration period.

- The left hand side in (65) can also be written as $\Delta(\mu B)$ so that once more, from $\Delta(\mu B) = \mu\Delta B$ follows $\Delta\mu = 0$, i.e., the magnetic moment is an invariant if **B** changes slowly on the gyration time scale.

E Show that the magnetic flux Φ "through a Larmor orbit" reads

$$\Phi = \frac{2\pi m}{q^2}\,\mu \qquad (67)$$

and is thus constant if μ is constant.

- The magnetic field lines are *adiabatically compressed* with increasing B (remember: $r_\mathrm{L} \sim 1/\sqrt{B}$). As $v_\perp$ increases for $\Delta B > 0$ according (65), this mechanism is used to heat plasmas.

2.6 Nonuniform E

- Let us study the $\mathbf{E}\times\mathbf{B}$-drift for nonuniform **E**. We take

$$\mathbf{E}(\mathbf{r}) = E_x(x)\mathbf{e}_x, \qquad \mathbf{B} = B\mathbf{e}_z \qquad (68)$$

so that

$$m\dot{v}_x = qv_yB + qE_x(x), \qquad m\dot{v}_y = -qv_xB, \qquad m\dot{v}_z = 0. \qquad (69)$$

- Expanding $E_x(x)$ around the guiding center using the unperturbed

  $$x - x_0 = r_L \sin \omega_c t$$

  we have

  $$E_x(x) = E_x(x_0) + \left.\frac{\partial E_x}{\partial x}\right|_{x_0} r_L \sin \omega_c t + \frac{1}{2} \left.\frac{\partial^2 E_x}{\partial x^2}\right|_{x_0} r_L^2 \sin^2 \omega_c t + \cdots . \quad (70)$$

  If L is a typical gradient length of $E_x(x)$ (or $K = 2\pi/L$ the corresponding wave number), we require $r_L/L \ll 1$ (or $Kr_L \ll 1$) for that series to be terminated after the quadratic term.

- For the time-averaged force we thus find

  $$\langle F_x \rangle = q\langle v_y \rangle B + q\langle E_x(x) \rangle = q \left[E_x(x_0) + \frac{1}{4} r_L^2 \left.\frac{\partial^2 E_x}{\partial x^2}\right|_{x_0} \right] . \quad (71)$$

- Hence, from (42) we expect a drift in y-direction, and more general

  $$\mathbf{v}_{gc} = \left(1 + \frac{1}{4} r_L^2 \nabla^2\right) \frac{\mathbf{E(r)} \times \mathbf{B}}{B^2} . \quad (72)$$

  This is called the $\mathbf{E} \times \mathbf{B}$-drift with *finite*-LARMOR-*radius* correction.

- As r_L is different for electrons and ions, the finite-LARMOR-radius correction will cause a different drift velocity for them. This is a qualitatively new feature (as compared to the pure $\mathbf{E} \times \mathbf{B}$-drift) and may lead to a *plasma instability*: charges separate $\rightarrow$ $\mathbf{E}$ increases $\rightarrow$ charges separate more $\rightarrow$ ...

- Note that the ∇B-drift above is also due to finite LARMOR-radii but $\sim K_B r_L$ (with K_B the wave number $\sim L_B^{-1}$ related to the gradient length L_B of the magnetic field). Now, for the nonuniform electric field, we find $\sim (Kr_L)^2$.

- The force-averaging procedure may appear a bit *ad hoc*. Hence, here comes an alternative approach. Starting point is again the equations of motion (69),

  $$m\dot{v}_x = qv_y B + qE_x(x), \qquad m\dot{v}_y = -qv_x B, \qquad m\dot{v}_z = 0.$$

  We make the ansatz

  $$v_x = v_x^{(0)}(\omega_c t) + v_x^{(d)}, \qquad v_y = v_y^{(0)}(\omega_c t) + v_y^{(d)}, \qquad x = x^{(0)}(\omega_c t) + x^{(d)}$$

where $v_{x,y}^{(0)}(\omega_c t)$ describes the unperturbed gyro-motion $\sim$ sin or $\cos\omega_c t$, and $v_{x,y}^{(d)}$ are the constant drift velocities we are looking for. Plugging this *separation of time-scales* ansatz into the equations of motion and averaging over a cyclotron period gives

$$0 = qv_y^{(d)}B + q\langle E_x[x^{(0)}(\omega_c t) + x^{(d)}]\rangle, \qquad 0 = -qv_x^{(d)}B,$$

because $\langle \dot{v}_{x,y}\rangle = \langle \dot{v}_{x,y}^{(0)}(\omega_c t)\rangle = 0$. Hence,

$$v_x^{(d)} = 0, \qquad v_y^{(d)} = -\frac{\langle E_x[x^{(0)}(\omega_c t) + x^{(d)}]\rangle}{B}.$$

Expanding the field around the drift position $x^{(d)}$ and using $x^{(0)}(\omega_c t) = r_L \sin\omega_c t$ leads to

$$v_y^{(d)} = -\left[E_x(x^{(d)}) + \frac{1}{4}r_L^2 \left.\frac{\partial^2 E_x}{\partial x^2}\right|_{x^{(d)}}\right]\frac{1}{B},$$

which is the same drift velocity we obtain from (72) for $\mathbf{E}(\mathbf{r}) = E_x(x)\mathbf{e}_x$.

2.7 POLARIZATION DRIFT

- Imagine a charge at rest in a magnetic field $\mathbf{B} = B\mathbf{e}_z$. As $\mathbf{v} = \mathbf{0}$ nothing happens. Now we switch-on an electric field, say in x-direction. The charge starts moving along the x-axis, v_x increases. Now, with finite $\mathbf{v}$, $\mathbf{B}$ is effective, and the particle starts gyrating. With the $\mathbf{E}$-field still on, there should be also an $\mathbf{E}\times\mathbf{B}$-drift. Note that previously we had a possibly spatially nonuniform but *time-independent* $\mathbf{E}$, and there was no drift in $\mathbf{E}$-direction!

- This is the new qualitative feature in this subsection: a drift $\parallel$ $\mathbf{E}$, the so-called *polarization drift*.

- We consider

$$\mathbf{E}(t) = E_x(t)\mathbf{e}_x, \qquad E_x(t) = E_{x0}\,\mathrm{e}^{\mathrm{i}\omega t} \qquad \Rightarrow \qquad \dot{E}_x = \mathrm{i}\omega E_x, \tag{73}$$

with no spatial dependence in E_x.

- Furthermore, we assume that E_x varies on a time-scale that is large compared to the cyclotron period,

$$\omega^2 \ll \omega_c^2. \tag{74}$$

- Similar to (69) we have

$$m\dot{v}_x = qv_yB + qE_x(\omega t), \qquad m\dot{v}_y = -qv_xB, \qquad m\dot{v}_z = 0. \tag{75}$$

- We want to separate the time scales again,

$$v_x = v_x^{(0)}(\omega_c t) + v_x^{(d)}(\omega t), \qquad v_y = v_y^{(0)}(\omega_c t) + v_y^{(d)}(\omega t). \tag{76}$$

  Plugging this ansatz into (75) and averaging over a cyclotron period yields

$$m\mathrm{i}\omega v_x^{(d)}(\omega t) = q v_y^{(d)}(\omega t) B + q E_x(\omega t), \qquad m\mathrm{i}\omega v_y^{(d)}(\omega t) = -q v_x^{(d)}(\omega t) B$$

  and thus

$$m\mathrm{i}\omega v_x^{(d)}(\omega t) = -q \frac{q v_x^{(d)}(\omega t) B}{m\mathrm{i}\omega} B + q E_x(\omega t)$$

$$\Rightarrow \; v_x^{(d)} = \underbrace{\left(1 + \frac{\omega_c^2}{(\mathrm{i}\omega)^2}\right)^{-1}}_{\simeq [\omega_c^2/(\mathrm{i}\omega)^2]^{-1}} \frac{q}{m} \frac{1}{\mathrm{i}\omega} E_x = \frac{q}{m} \frac{\mathrm{i}\omega}{\omega_c^2} E_x = \pm \frac{1}{\omega_c B} \partial_t E_x$$

  or

$$\mathbf{v}_{\mathrm{gc,pol}} = \pm \frac{1}{\omega_c B} \frac{\partial}{\partial t} \mathbf{E}. \tag{77}$$

- This is the drift in the direction of **E** we anticipated in the beginning of this section. It is only effective if the temporal change in the electric field is fast enough (as it goes with ω/ω_c). However, for our derivation we assumed that it is not too fast, because $\omega^2 \ll \omega_c^2$. Unless this condition is fulfilled, a separation of time-scales does not make sense.

- The polarization effect is similar to the case of solid dielectrics but not quite the same. Imagine a solid dielectric inside a capacitor to which a voltage of variable frequency can be applied. The dipoles will orient themselves according to the electric field, creating the polarization. This works for a static field ($\omega = 0$) as well as for $\omega \neq 0$. If $\omega \ll \Omega$, where Ω is some internal frequency with which the dipoles would freely oscillate, the dipoles can easily follow. Now imagine a plasma between the capacitor plates. Without additional magnetic field the plasma would simply serve as a conducting medium, and a DC ($\omega = 0$) or AC ($\omega \neq 0$) current would set in. With a perpendicular magnetic field we do not have dipoles but charges that are forced to perform circular orbits around the magnetic field. In the AC case the net effect is similar to what happens in the solid dielectric because charges are displaced on a length scale of the order of r_L. However, the polarization effect is absent in the static field case (where only the $\mathbf{E} \times \mathbf{B}$-drift occurs).

2.8 ADIABATIC INVARIANTS

- Consider a one-dimensional system described by a Hamiltonian $H(p, q; \Omega)$ with Ω a parameter that changes "slowly", e.g.,

$$\left| \frac{\frac{\mathrm{d}\Omega}{\mathrm{d}(t/T)}}{\Omega} \right| = T \left| \frac{\frac{\mathrm{d}\Omega}{\mathrm{d}t}}{\Omega} \right| \ll 1. \tag{78}$$

  Here, T is the period of the motion for fixed Ω and a given total energy E. Hence we assume that for fixed Ω the system undergoes periodic motion of period T. Then, for *adiabatically* changing Ω, we expect the motion to be *quasi-periodic*, i.e., the orbits in phase space will not exactly be closed, but "almost".

- One can show[2] that

$$\left\langle \frac{\mathrm{d}I}{\mathrm{d}t} \right\rangle = 0 \tag{79}$$

  for

$$I = \oint p \, \mathrm{d}q \tag{80}$$

  where p and q are canonical momentum and position, respectively. Further,

$$\langle f \rangle = \frac{1}{T} \int_0^T f(t) \, \mathrm{d}t, \tag{81}$$

  and the integral over a closed path in (80) is meant to be taken for the given E and the current Ω.

- The rigorous theory behind adiabatic invariants is rich, interesting, and subtle.[3] One could spend a whole lecture on it. However, in this lecture we merely apply the theory.

- In general, for higher-dimensional systems which are *non-separable* adiabatic invariants do not exist. However, we will see below that, e.g., in the case of a charged particle in the (admittedly idealized) magnetic field of the earth there are three very distinct time-scales so that the dynamics is quasi-separable. One can *adiabatically eliminate* step by step the respective fastest time-scale.

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[2] See any good textbook on Classical Mechanics.

[3] See, e.g., V.I. Arnold, *Mathematical Methods of Classical Mechanics* or A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*.

- As an example, consider a harmonic oscillator whose frequency $\Omega(t)$ changes slowly compared to the period $T = 2\pi/\Omega$,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\Omega^2(t)q^2.$$

For a given energy E the trajectory in phase space is an ellipse with semi-axes $\sqrt{2mE}$ and $\sqrt{2E/m\Omega^2}$. Hence we obtain for (80)

$$I = \pi\sqrt{2mE}\sqrt{2E/m\Omega^2} = \text{const} \quad \Rightarrow \quad \frac{E(t)}{\Omega(t)} = \text{const},$$

i.e., $E(t)$ and $\Omega(t)$ change adiabatically such that their ratio is constant.

2.8.1 *First adiabatic invariant*

- Now let us calculate the adiabatic invariant related to the gyro-motion of a charged particle about the magnetic field lines. The Hamiltonian reads

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} = \frac{(\mathbf{p}_\perp - q\mathbf{A}_\perp)^2}{2m} + \frac{(p_\parallel - qA_\parallel)^2}{2m}. \tag{82}$$

- For the cyclotron motion We can treat the magnetic field locally constant $\mathbf{B} = B\mathbf{e}_\parallel$. We can thus choose for the vector potential, e.g.,

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \quad \Rightarrow A_\parallel = 0. \tag{83}$$

- Because

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A} \tag{84}$$

we have to evaluate

$$I_1 = \oint \mathbf{p} \cdot \mathrm{d}\mathbf{q} = \oint m\mathbf{v} \cdot \mathrm{d}\mathbf{r} + \oint q\mathbf{A} \cdot \mathrm{d}\mathbf{r} = mv_\perp 2\pi r_\mathrm{L} + \int q\nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{n} \tag{85}$$

where the last integral is over the area enclosed by the gyro orbit, d**n** being the (properly oriented) surface normal vector element. We can write this as

$$I_1 = 2\pi\frac{mv_\perp^2}{\omega_\mathrm{c}} + \int q\mathbf{B} \cdot \mathrm{d}\mathbf{n}.$$

As d**n** is anti-parallel to **B** for a positive charge and parallel for a negative the result is in any case

$$I_1 = 2\pi\frac{mv_\perp^2}{\omega_\mathrm{c}} - |q|B\pi r_\mathrm{L}^2 = 2\pi\frac{|q|B}{m}\frac{mv_\perp^2}{\omega_\mathrm{c}^2} - |q|B\pi\frac{v_\perp^2}{\omega_\mathrm{c}^2} = \pi\frac{|q|B}{m}\frac{mv_\perp^2}{\omega_\mathrm{c}^2} = \pi m\frac{mv_\perp^2}{|q|B}$$

so that

$$I_1 = 2\pi \frac{m}{|q|} \mu = \text{const}, \tag{86}$$

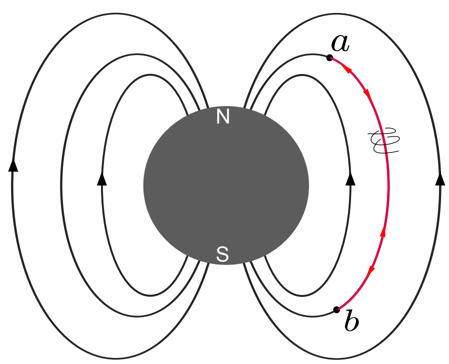
with the magnetic moment $\mu = mv_\perp^2/2B$ [see (54)], is an adiabatic invariant.

- Hence, we confirm the previous result that the magnetic moment μ itself is constant,

$$I_1 \sim \mu = \text{adiabatic invariant}. \tag{87}$$

2.8.2 *Second adiabatic invariant*

- Having in mind the motion of a charged particle (e.g., a solar-wind proton) in the earth's magnetosphere, we expect another adiabatic invariant related to the north-south bouncing due to the magnetic-mirror effect along the red line in the figure:



- Averaging-out the gyro-motion about the magnetic field lines, what remains is the Hamiltonian for the guiding center

$$H_1 = \langle H \rangle_{2\pi/\omega_c} = \mu B(q_\parallel) + \frac{(p_\parallel - qA_\parallel)^2}{2m}. \tag{88}$$

E Show (88) explicitly, i.e., plug the gyro-motion orbits into H (82) and average.

- Hence, the adiabatic invariant connected to the nort-south bouncing is

$$I_2 = \oint p_\parallel \, \mathrm{d}q_\parallel = m \oint v_\parallel \, \mathrm{d}q_\parallel + q \oint A_\parallel \, \mathrm{d}q_\parallel.$$

This time we cannot set $A_\parallel = 0$ because without any $A_\parallel$ the magnetic field would be constant in magnitude and thus there would be no bouncing. However, the integral $\oint A_\parallel \, \mathrm{d}q_\parallel$ vanishes.

- From energy conservation, employing that numerically $\frac{(p_\parallel - qA_\parallel)^2}{2m} = \frac{1}{2}mv_\parallel^2$, we find

$$E = \mu B(q_\parallel) + \frac{1}{2}mv_\parallel(q_\parallel)^2. \tag{89}$$

- Let $q_\parallel = a$ and $q_\parallel = b$ be the turning points defined by $v_\parallel = 0$,

$$E = \mu B(a) = \mu B(b). \tag{90}$$

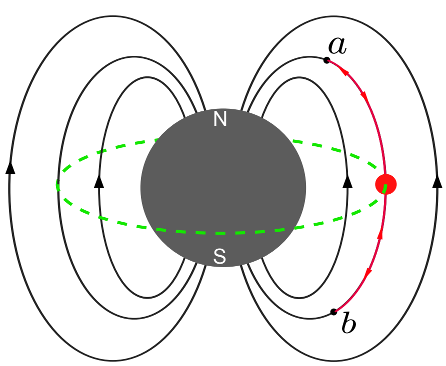
- We can then write

$$I_2 = 2m \int_a^b v_\parallel \, \mathrm{d}q_\parallel = \text{adiabatic invariant}. \tag{91}$$

- If the field configuration changes slowly on the time-scale of this bouncing motion I_2 will be conserved.

2.8.3 *Third adiabatic invariant*

- The curvature and ∇B drift discussed above would lead to another periodic motion of the "bouncing center" along the green, dashed circle in the following figure if the earth's magnetic field were ideally dipole-like.



- Averaging-out the bouncing motion leads to an average position at the equator (red dot) that drifts equatorially around the earth.

E Which effect has the earth's rotation?

- If the magnetic field were deformed on a slow time scale compared to the unperturbed equatorially drift around the earth there would be another adiabatic invariant I_3 related to that motion. However, fluctuations in the magnetosphere are not really slow on that time scale so that I_3 is not constant. Nevertheless it is instructive to calculate it.

- The Hamiltonian reads (again)

$$H_2 = \frac{[\mathbf{p}_\circ - q\mathbf{A}(\mathbf{q}_\circ)]^2}{2m} \tag{92}$$

  but now $\mathbf{q}_\circ$ and $\mathbf{p}_\circ = m\mathbf{v}_{\mathrm{curv},\nabla B} + q\mathbf{A}(\mathbf{q}_\circ)$ are the canonical coordinates describing the equatorial drift motion.

- In analogy to (85) we find

$$I_3 = 2\pi R\, m\, v_{\mathrm{curv},\nabla B} + q \int \mathbf{B} \cdot \mathrm{d}\mathbf{n}. \tag{93}$$

- The first term is numerically small compared to the second, which is just the magnetic flux Φ through the area enclosed by the equatorial orbit around the earth. Hence,

$$I_3 = q\Phi = \text{adiabatic invariant}. \tag{94}$$

- If the magnetic field was slowly distorted by, e.g., the solar wind, the gyro- and bouncing-motion averaged orbit would change such that the total enclosed magnetic flux stays constant. Although this is not valid for particle motion in the magnetosphere it may be useful under other circumstances, e.g., for magnetic bottles.

- An example from Kulsrud's book: 10-keV proton 5 earth radii away from the earth $\Rightarrow \omega_c = 24\,\mathrm{s}^{-1}$, $v_\perp = 1.6 \times 10^6\,\mathrm{m/s}$, $r_L = 67\,\mathrm{km}$. For a pitch angle $\arctan v_\perp/v_\parallel = 45°$ the proton moves $\pm 6000\,\mathrm{km}$ away from the equator along the field line, and the bouncing period is $\simeq 100\,\mathrm{s}$. The equatorial drift velocity is "only" $6\,\mathrm{km/s}$ so that it takes $3.6 \times 10^4\,\mathrm{s}$ to circle the earth.

  Note the three nicely separated time-scales in the problem:

$$T_{\mathrm{cyclotron}} = 0.26 \ll T_{\mathrm{bounce}} = 100 \ll T_{\mathrm{equatorial}} = 3.6 \times 10^4\,\mathrm{s}.$$

3 KINETIC DESCRIPTION OF PLASMA I

- In this chapter we will derive the equations governing plasma from first principles. We follow a ”top-down” approach and show how from quite general statistical considerations the Vlasov equation can be derived as long as “collisions” between plasma particles are negligible, i.e., collective effects dominate. Later, in the chapter “Kinetic Description of Plasma II” we will discuss how collisions can be included.

3.1 BBGKY hierarchy

- We consider, for simplicity, a classical, nonrelativistic *one component plasma* of N identical particles with charges q and masses m in a volume V, e.g., electrons and a homogeneous, smeared-out background of ionic charge density $qn = qN/V$ in order to render the total system neutral. At the end of this section we generalize the result to multi-component plasma.

- As a starting point for the derivation of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy the Klimontovich distribution function

$$K(X,t) = \sum_{i=1}^{N} \delta[X - X_i(t)] \tag{95}$$

  may be used. Here, $X_i(t) = (\mathbf{r}_i(t), \mathbf{v}_i(t))$ collects the phase space coordinates of particle i, and $X = (\mathbf{r}, \mathbf{v})$. Note that we write $K(X,t)$ just to avoid the more clumsy $K(X, \{X_i(t)\})$. The explicit time-dependence arises because the solutions $X_i(t)$ of the equations of motion of all particles are thought of being plugged into K.

- Written in terms of $\mathbf{r}$ and $\mathbf{v}$ we have

$$K(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_i(t)]\, \delta[\mathbf{v} - \mathbf{v}_i(t)]. \tag{96}$$

- We see that $K(X,t)$ is a phase space distribution which contains the full knowledge about the classical system: all positions and all momenta.

- One may think of K as N delta-peaks, moving around in 6D phase space.

E Show that from the classical equations of motion

$$\frac{\partial K}{\partial t} + \dot{X} \cdot \frac{\partial K}{\partial X} = 0 = \frac{\mathrm{d}K}{\mathrm{d}t} \tag{97}$$

follows.

- Explicitly written in $\mathbf{r}$ and $\mathbf{v}$ we have

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} \right) K(\mathbf{r}, \mathbf{v}, t) = 0, \tag{98}$$

which may be called the Klimontovich or the Klimontovich-Dupree equation.

- In plasma the dominant force is the Lorentz force so that

$$\dot{\mathbf{v}} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{99}$$

where the fields $\mathbf{E}$ and $\mathbf{B}$ may be split into an external part $\mathbf{E}_0$, $\mathbf{B}_0$ describing the externally applied fields, and the fields $\mathbf{e}$, $\mathbf{b}$ generated by the plasma particles themselves. The latter can be in principle calculated via the charge and the current density

$$\rho(\mathbf{r}, t) = q \left[\int \mathrm{d}^3 v \, K(X, t) - n \right], \tag{100}$$

$$\mathbf{j}(\mathbf{r}, t) = q \int \mathrm{d}^3 v \, \mathbf{v} K(X, t) \tag{101}$$

from Maxwell's equations

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho, \tag{102}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{103}$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \tag{104}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \dot{\mathbf{E}}. \tag{105}$$

- In the so-called *electrostatic approximation*, in which magnetic-field, retardation, and relativistic effects are ignored we have

$$\mathbf{b}(\mathbf{r}, t) = \mathbf{0}, \qquad \mathbf{e}(\mathbf{r}, t) = -\nabla_{\mathbf{r}} \Phi(\mathbf{r}, t) \tag{106}$$

with the scalar potential

$$\Phi(\mathbf{r}, t) = \int \mathrm{d}^3 r' \, \frac{q \int \mathrm{d}^3 v' \, K(X', t)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} = \frac{q}{4\pi\epsilon_0} \int \mathrm{d}X' \, \frac{K(X', t)}{|\mathbf{r} - \mathbf{r}'|}. \tag{107}$$

- The homogeneous charge background does not contribute to $\mathbf{e}(\mathbf{r}, t)$.
- We can then write the KLIMONTOVICH equation (98) in the form

$$\left(\partial_t + L(X,t) - \frac{1}{n}\int \mathrm{d}X'\, V(X,X',t)K(X',t)\right) K(X,t) = 0, \tag{108}$$

  where

$$L(X,t) = \mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{q}{m}\left(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0\right)\cdot\nabla_{\mathbf{v}} \tag{109}$$

  is a single-particle operator, and

$$V(X,X',t) = \frac{nq^2}{4\pi\epsilon_0 m}\left(\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\nabla_{\mathbf{v}} \tag{110}$$

  is a two-particle operator which arises because of the COULOMB interaction.

- The KLIMONTOVICH equation (108) describes the microscopic evolution of the microscopic, "fine-grained", classical distribution function $K(X,t)$, which is not useful for practical purposes.
- We need to make a connection to the more "coarse-grained" description of a plasma in terms of a single-particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$ or macroscopic entities such as density $\rho(\mathbf{r}, t)$, pressure $p(\mathbf{r}, t)$, temperature $T(\mathbf{r}, t)$ etc.
- The connection is established in a statistical way. A microscopic state of the system is described by a point in $6N$-dimensional phase space[1] Γ,

$$\{X_i\} = (X_1, X_2, \ldots, X_N). \tag{111}$$

- In Statistical Physics you have been introduced to the concept of *statistical ensembles* (although for thermal equilibrium mostly). The statistical operator, density operator, or, classically, the statistical distribution function was denoted ρ there. In order not to confuse it with charge densities we call it $D(\{X_i\}, t)$ here. Remember, D weights microscopic states according to our knowledge about the system. We assume that D is normalized to unity,

$$\int \mathrm{d}\{X_i\}\, D(\{X_i\}, t) = 1.$$

[1] Note that the KLIMONTOVICH distribution "lives" in 6-dimensional phase space.

- The statistical distribution function fulfills the LIOUVILLE equation

$$\partial_t D + \{\dot{X}_i\} \cdot \partial_{\{X_i\}} D = 0, \tag{112}$$

where $\partial_{\{X_i\}}$ runs over all $6N$ phase space coordinates.

- Consider a fine-grained quantity

$$A(X, X', \dots, \{X_i(t)\}, t).$$

The KLIMONTOVICH distribution is such a quantity (we just suppressed the dependence on the individual single-particle phase-space points $\{X_i\}$ in the argument). Expectation values of such quantities are calculated according

$$\langle A(X, X', \dots, t)\rangle = \int \mathrm{d}\{X_i(t)\}\, A(X, X', \dots, \{X_i(t)\}, t)\, D(\{X_i(t)\}, t).$$

- Since the $\{X_i(t)\}$ in the fine-grained quantity A evolve in time just like the respective $\{X_i(t)\}$ in D one may do the actual weighting at any time, e.g., at $t = 0$, and let A evolve thereafter,

$$\langle A(X, X', \dots, t)\rangle = \int \mathrm{d}\{X_i(0)\}\, A(X, X', \dots, \{X_i(\{X_i(0)\}, t)\}\, D(\{X_i(0)\}, 0).$$

Here, $\{X_i(\{X_i(0)\}, t)\}$ indicates the Γ-point $\{X_i(t)\}$ that was at $t = 0$ at $\{X_i(0)\}$.

- For the KLIMONTOVICH distribution we obtain

$$\begin{aligned}
\langle K(X,t)\rangle &= \int \mathrm{d}\{X_i(t)\}\, K(X, \{X_i(t)\})\, D(\{X_i(t)\}, t) \\
&= \int \mathrm{d}\{X_i(t)\} \sum_{i=1}^{N} \delta[\mathrm{X} - \mathrm{X}_i(t)]\, D(\{X_i(t)\}, t) \\
&= \int \mathrm{d}X_2(t) \cdots \mathrm{d}X_N(t) D(X, X_2(t), \dots, X_N(t), t) \\
&\quad + \int \mathrm{d}X_1(t)\mathrm{d}X_3(t) \cdots \mathrm{d}X_N(t) D(X_1(t), X, X_3(t), \dots, X_N(t), t) + \cdots \\
&= N \underbrace{\int \mathrm{d}X_2(t) \cdots \mathrm{d}X_N(t) D(X, X_2(t), \dots, X_N(t), t)}_{D_1(X,t)}
\end{aligned}$$

where we assume that D is invariant under the permutation of particles.[2] D_1 is the single-particle distribution function normalized to unity, $\int \mathrm{d}X\, D_1(X,t) = 1$.

Hence,

$$\langle K(X, \dots, t)\rangle = N D_1(X, t). \tag{113}$$

[2] We consider here one particular species of indistinguishable particles. In general, a plasma consists of several species of indistinguishable particles.

- For a product of Klimontovich distributions we obtain

$$\begin{aligned} \langle K(X,t)K(X',t)\rangle &= \int \mathrm{d}\{X_i(t)\}\, K(X,\{X_i(t)\})K(X',\{X_i(t)\})\, D(\{X_i(t)\},t) \\ &= \int \mathrm{d}\{X_i(t)\}\, \sum_{i=1}^{N} \delta[X-X_i(t)] \sum_{j=1}^{N} \delta[X'-X_j(t)]\, D(\{X_i(t)\},t) \\ &= \int \mathrm{d}\{X_i(t)\}\, \sum_{i=1}^{N} \delta[X-X_i(t)]\delta[X'-X_i(t)]\, D(\{X_i(t)\},t) \\ &\quad + \int \mathrm{d}\{X_i(t)\}\, \sum_{i\neq j}^{N} \delta[X-X_i(t)]\delta[X'-X_j(t)]\, D(\{X_i(t)\},t) \\ &= \delta(X-X')ND_1(X,t) + N(N-1)D_2(X,X',t) \end{aligned} \tag{114}$$

with

$$D_2(X,X',t) = \int \mathrm{d}X_3(t)\cdots \mathrm{d}X_N(t) D(X,X',X_3(t),\ldots,X_N(t),t) \tag{115}$$

the two-particle distribution function normalized to unity, $\int \mathrm{d}X\mathrm{d}X'\, D_2(X,X',t) = 1$.

- The *single-particle distribution function* $f_1(X,t)$ is defined[3] as

$$f_1(X,t) = \frac{1}{n}\langle K(X,t)\rangle = VD_1(X,t), \qquad n = \frac{N}{V}. \tag{116}$$

- Now consider the product of two Klimontovich distributions

$$\begin{aligned} K(X,\{X_i(t)\})K(X',\{X_j(t)\}) &= \sum_{i=1}^{N} \delta[X-X_i(t)] \sum_{j=1}^{N} \delta[X'-X_j(t)] \\ &= \delta(X-X') \sum_{i=1}^{N} \delta[X-X_i(t)] \\ &\quad + \sum_{i\neq j}^{N} \delta[X-X_i(t)]\delta[X'-X_j(t)]. \end{aligned} \tag{117}$$

Taking the ensemble average gives, using (116),

$$\langle K(X,t)K(X',t)\rangle = \delta(X-X')nf_1(X,t) + n^2 f_2(X,X',t),$$

i.e., the definition of the *two-particle distribution function*

$$f_2(X,X',t) = \frac{1}{n^2}\langle K(X,t)K(X',t)\rangle - \delta(X-X')\frac{1}{n}f_1(X,t). \tag{118}$$

[3] For the sake of confusion, distribution functions with different normalizations are used in the Plasma Physics literature. Here, $\int \mathrm{d}X\, f_1(X,t) = V$.

- Comparison with (114) shows that

$$f_2(X,X',t) = \frac{N(N-1)}{n^2} D_2(X,X',t) \simeq V^2 D_2(X,X',t) \tag{119}$$

for large N.

E Similarly

$$\begin{aligned} f_3(X,X',X'',t) &= \frac{1}{n^3}\langle K(X,t)K(X',t)K(X'',t)\rangle \\ &\quad -\frac{1}{n^2}\delta(X-X')\delta(X-X'')f_1(X,t) \\ &\quad -\frac{1}{n}[\delta(X-X')f_2(X',X'',t)+\delta(X'-X'')f_2(X'',X,t) \\ &\qquad +\delta(X''-X)f_2(X,X',t)] \\ &\simeq V^3 D_3(X,X',X'',t) \end{aligned} \tag{120}$$

with $D_3(X,X',X'',t) = \int \mathrm{d}X_4(t)\cdots \mathrm{d}X_N(t) D(X,X',X'',X_4(t),\ldots,X_N(t),t)$.

E How are the distribution functions f_i normalized?

- Now we are prepared to derive the BBGKY hierarchy. We write the KLIMONTOVICH equation (108) in the form (suppressing the time arguments from now on)

$$[\partial_t + L(1)]\, K(1) = \frac{1}{n}\int \mathrm{d}2\, V(1,2)K(1)K(2) \tag{121}$$

where we write $1,2,3,\ldots$ instead of $X,X',X'',\ldots$

- Ensemble-averaging yields, using (118),

$$[\partial_t + L(1)]\, nf_1(1) = \int \mathrm{d}2\, V(1,2)[nf_2(1,2)+\delta(1-2)f_1(1)].$$

- Because

$$\begin{aligned} &\int \mathrm{d}2\, V(1,2)\delta(1-2)f_1(1) \\ &= \xi\int \mathrm{d}^3r'\int \mathrm{d}^3v'\, \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\nabla_{\mathbf{v}}\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}')f_1(\mathbf{r},\mathbf{v},t), \qquad \xi = \frac{nq^2}{4\pi\epsilon_0 m} \\ &= \xi\int \mathrm{d}^3r' \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\delta(\mathbf{r}-\mathbf{r}')\nabla_{\mathbf{v}}f_1(\mathbf{r},\mathbf{v},t) = \xi\nabla_{\mathbf{v}}f_1(\mathbf{r},\mathbf{v},t)\cdot\int_{\mathbb{R}^3}\mathrm{d}^3u\, \underbrace{\frac{\mathbf{u}}{|u|^3}}_{\text{odd}}\underbrace{\delta(\mathbf{u})}_{\text{even}} \end{aligned}$$

$$= \quad 0 \tag{122}$$

this reduces to

$$[\partial_t + L(1)]\, f_1(1) = \int \mathrm{d}2\, V(1,2) f_2(1,2). \tag{123}$$

E Show that from the equation of motion for a product of KLIMONTOVICH distribution functions one finds—upon ensemble averaging—

$$\begin{aligned}\left[\partial_t + L(1) + L(2) - \frac{1}{n}\left\{V(1,2) + V(2,1)\right\}\right] f_2(1,2) \\ = \int \mathrm{d}3\, [V(1,3) + V(2,3)] f_3(1,2,3).\end{aligned} \tag{124}$$

S One must pay attention to the fact that both $L(i)$ and $V(i,j)$ contain differential operators acting with respect to variables comprised in i.

The time derivative of a product of two KLIMONTOVICH distribution functions reads, using (108),

$$\begin{aligned}\partial_t[K(1)K(2)] &= [\partial_t K(1)]K(2) + K(1)\partial_t K(2) \\ &= K(2)\left[-L(1) + \frac{1}{n}\int \mathrm{d}3\, V(1,3)K(3)K(1)\right] + K(1)\left[-L(2) + \frac{1}{n}\int \mathrm{d}3\, V(2,3)K(3)\right] K(2) \\ &= -[L(1) + L(2)]K(1)K(2) + \frac{1}{n}\int \mathrm{d}3\, [V(1,3) + V(2,3)]K(1)K(2)K(3).\end{aligned}$$

Upon ensemble averaging we obtain with (118) and (120)

$$\begin{aligned}\partial_t\langle K(1)K(2)\rangle &= \partial_t[n^2 f_2(1,2) + n\delta(1-2) f_1(1)], \\ -[L(1)+L(2)]\langle K(1)K(2)\rangle &= -[L(1)+L(2)][n^2 f_2(1,2) + n\delta(1-2) f_1(1)], \\ \frac{1}{n}\int \mathrm{d}3\, [V(1,3)+V(2,3)]\langle K(1)K(2)K(3)\rangle & \\ = \frac{1}{n}\int \mathrm{d}3\, [V(1,3)+V(2,3)][n^3 f_3(1,2,3) &+ n\delta(1-2)\delta(1-3) f_1(1) \\ &+ n^2\{\delta(1-2) f_2(2,3) + \delta(2-3) f_2(3,1) + \delta(3-1) f_2(1,2)\}].\end{aligned} \tag{125}$$

We first show that $\int \mathrm{d}3\, [V(1,3) + V(2,3)]\delta(1-2)\delta(1-3) f_1(1)$ vanishes:

$$\begin{aligned}&\int \mathrm{d}3\, [V(1,3) + V(2,3)]\delta(1-2)\delta(1-3) f_1(1) \\ &= \xi \int \mathrm{d}^3 r_3 \int \mathrm{d}^3 v_3 \left[\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_1} + \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_2}\right] \\ &\quad \times \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{v}_1 - \mathbf{v}_2)\delta(\mathbf{r}_1 - \mathbf{r}_3)\delta(\mathbf{v}_1 - \mathbf{v}_3) f_1(\mathbf{r}_1, \mathbf{v}_1, t) \\ &= \xi \int \mathrm{d}^3 r_3 \left[\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_1} + \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_2}\right] \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{v}_1 - \mathbf{v}_2)\delta(\mathbf{r}_1 - \mathbf{r}_3) f_1(\mathbf{r}_1, \mathbf{v}_1, t)\end{aligned}$$

$$\begin{aligned}
&= \xi\delta(\mathbf{r}_1-\mathbf{r}_2)\nabla_{\mathbf{v}_1}[\delta(\mathbf{v}_1-\mathbf{v}_2)f_1(\mathbf{r}_1,\mathbf{v}_1,t)]\cdot\int \mathrm{d}^3r_3\,\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}\delta(\mathbf{r}_1-\mathbf{r}_3)\\
&\quad+\xi\delta(\mathbf{r}_1-\mathbf{r}_2)f_1(\mathbf{r}_1,\mathbf{v}_1,t)[\nabla_{\mathbf{v}_2}\delta(\mathbf{v}_1-\mathbf{v}_2)]\cdot\int \mathrm{d}^3r_3\,\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}\delta(\mathbf{r}_2-\mathbf{r}_3)\\
&= 0.
\end{aligned}$$

The last step follows for the same symmetry argument as in (122).

Next we find

$$\begin{aligned}
[L(1)+L(2)]\delta(1-2)f_1(1) &= f_1(1)L(1)\delta(1-2)+\delta(1-2)L(1)f_1(1)+f_1(1)L(2)\delta(1-2)\\
&= \delta(1-2)L(1)f_1(1)
\end{aligned}$$

because

$$\begin{aligned}
f_1(1)L(1)\delta(1-2) &= f_1(\mathbf{r}_1,\mathbf{v}_1,t)[\mathbf{v}_1\cdot\nabla_{\mathbf{r}_1}+\mathbf{a}_1\cdot\nabla_{\mathbf{v}_1}]\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)\\
&= -f_1(\mathbf{r}_1,\mathbf{v}_1,t)[\mathbf{v}_1\cdot\nabla_{\mathbf{r}_2}+\mathbf{a}_1\cdot\nabla_{\mathbf{v}_2}]\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)\\
&= -f_1(\mathbf{r}_1,\mathbf{v}_1,t)[\mathbf{v}_2\cdot\nabla_{\mathbf{r}_2}+\mathbf{a}_2\cdot\nabla_{\mathbf{v}_2}]\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)\\
&= -f_1(1)L(2)\delta(1-2),
\end{aligned}$$

where $\mathbf{a}_i = \frac{q}{m}[\mathbf{E}_0(\mathbf{r}_i,t)+\mathbf{v}_i\times\mathbf{B}_0(\mathbf{r}_i,t)]$.

Moreover

$$\begin{aligned}
&\int \mathrm{d}3\,[V(1,3)+V(2,3)]\delta(1-2)f_2(2,3)\\
&\quad= \xi\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}\cdot\nabla_{\mathbf{v}_1}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\qquad+\xi\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}\cdot\nabla_{\mathbf{v}_2}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\quad= \xi\nabla_{\mathbf{v}_1}\delta(\mathbf{v}_1-\mathbf{v}_2)\cdot\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}\delta(\mathbf{r}_1-\mathbf{r}_2)f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\qquad+\xi[\nabla_{\mathbf{v}_2}\delta(\mathbf{v}_1-\mathbf{v}_2)]\cdot\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}\delta(\mathbf{r}_1-\mathbf{r}_2)f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\qquad+\xi\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}\cdot\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)\nabla_{\mathbf{v}_2}f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\quad= \xi\nabla_{\mathbf{v}_1}\delta(\mathbf{v}_1-\mathbf{v}_2)\cdot\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}\delta(\mathbf{r}_1-\mathbf{r}_2)f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\qquad-\xi[\nabla_{\mathbf{v}_1}\delta(\mathbf{v}_1-\mathbf{v}_2)]\cdot\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}\delta(\mathbf{r}_1-\mathbf{r}_2)f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\qquad+\xi\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}\cdot\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)\nabla_{\mathbf{v}_2}f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\quad= \xi\int \mathrm{d}^3r_3\int \mathrm{d}^3v_3\,\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3}\cdot\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{v}_1-\mathbf{v}_2)\nabla_{\mathbf{v}_2}f_2(\mathbf{r}_2,\mathbf{r}_3,\mathbf{v}_2,\mathbf{v}_3,t)\\
&\quad= \delta(1-2)\int \mathrm{d}3\,V(2,3)f_2(2,3).
\end{aligned}$$

Multiplying (123) by $n\delta(1-2)$ gives

$$n\delta(1-2)[\partial_t+L(1)]f_1(1) = n\delta(1-2)\int \mathrm{d}3\,V(2,3)f_2(2,3).$$

Hence, putting all surviving terms in (125) together we obtain

$$\begin{aligned}
[\partial_t + L(1) + L(2)] f_2(1,2) &= \int \mathrm{d}3\, [V(1,3) + V(2,3)] f_3(1,2,3) \\
&\quad + \frac{1}{n} \int \mathrm{d}3\, [V(1,3) + V(2,3)] [\delta(2-3) f_2(3,1) + \delta(3-1) f_2(1,2)] \\
&= \int \mathrm{d}3\, [V(1,3) + V(2,3)] f_3(1,2,3) \\
&\quad + \frac{1}{n} \{V(1,2) f_2(2,1) + V(2,1) f_2(1,2)\} \\
&\quad + \frac{1}{n} \int \mathrm{d}3\, V(1,3)\delta(3-1) f_2(1,2) + \frac{1}{n} \int \mathrm{d}3\, V(2,3)\delta(2-3) f_2(3,1).
\end{aligned}$$

The last two terms vanish because of symmetry again:

$$\begin{aligned}
&\int \mathrm{d}3\, V(1,3)\delta(3-1) f_2(1,2) + \frac{1}{n} \int \mathrm{d}3\, V(2,3)\delta(2-3) f_2(3,1) \\
&= \xi \int \mathrm{d}^3 r_3 \int \mathrm{d}^3 v_3 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_1} \delta(\mathbf{r}_3 - \mathbf{r}_1)\delta(\mathbf{v}_3 - \mathbf{v}_1) f_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2, t) \\
&\quad + \xi \int \mathrm{d}^3 r_3 \int \mathrm{d}^3 v_3 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_2} \delta(\mathbf{r}_2 - \mathbf{r}_3)\delta(\mathbf{v}_2 - \mathbf{v}_3) f_2(\mathbf{r}_3, \mathbf{r}_1, \mathbf{v}_3, \mathbf{v}_1, t) \\
&= \xi \int \mathrm{d}^3 r_3\, \delta(\mathbf{r}_3 - \mathbf{r}_1) \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_1} f_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2, t) \\
&\quad + \xi \int \mathrm{d}^3 r_3\, \delta(\mathbf{r}_2 - \mathbf{r}_3) \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \cdot \nabla_{\mathbf{v}_2} f_2(\mathbf{r}_3, \mathbf{r}_1, \mathbf{v}_2, \mathbf{v}_1, t) \\
&= \xi \nabla_{\mathbf{v}_1} f_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2, t) \cdot \int \mathrm{d}^3 r_3\, \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \delta(\mathbf{r}_3 - \mathbf{r}_1) \\
&\quad + \xi \nabla_{\mathbf{v}_2} f_2(\mathbf{r}_2, \mathbf{r}_1, \mathbf{v}_2, \mathbf{v}_1, t) \cdot \int \mathrm{d}^3 r_3\, \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \delta(\mathbf{r}_2 - \mathbf{r}_3) = 0.
\end{aligned}$$

Then, with $f_2(2,1) = f_2(1,2)$ we finally have

$$\begin{aligned}
[\partial_t + L(1) + L(2)] f_2(1,2) &= \int \mathrm{d}3\, [V(1,3) + V(2,3)] f_3(1,2,3) \\
&\quad + \frac{1}{n} [V(1,2) + V(2,1)]\, f_2(1,2),
\end{aligned}$$

which is the desired result.

- Similarly the equation of motion for f_3 couples to f_4 etc. This is the *BBGKY hierarchy*, named after Bogoliubov, Born, Green, Kirkwood and Yvon, which can be summarized in the form

$$\boxed{\begin{aligned}
&\left[\partial_t + \sum_{i=1}^{s} L(i) - \frac{1}{n} \sum_{i \neq j}^{s} V(i,j)\right] f_s(1, \ldots, s) \\
&\qquad = \sum_{i=1}^{s} \int \mathrm{d}(s+1)\, V(i, s+1) f_{s+1}(1, 2, \ldots, s+1)
\end{aligned}} . \tag{126}$$

3.2 VLASOV EQUATION

- We write the distribution functions in terms of *correlation functions* C_s,

$$\begin{aligned} C_1(1) &= f_1(1), && (127)\\ C_2(1,2) &= f_2(1,2) - C_1(1)C_1(2), && (128)\\ C_3(1,2,3) &= f_3(1,2,3) - C_1(1)C_1(2)C_1(3) && (129)\\ &\quad - C_1(1)C_2(2,3) - C_1(2)C_2(3,1) - C_1(3)C_2(1,2), &&\\ &\vdots && \end{aligned}$$

- In particular, with

$$F(1) = f_1(1), \tag{130}$$

and the *pair correlation function*

$$G(1,2) = C_2(1,2) \tag{131}$$

we have

$$f_2(1,2) = F(1)F(2) + G(1,2). \tag{132}$$

- The first equation of the BBGKY hierarchy (126), i.e., the one for $s = 1$, reads

$$\left[\partial_t + L(1) - \int \mathrm{d}2\, V(1,2)F(2)\right] F(1) = \int \mathrm{d}2\, V(1,2)G(1,2). \tag{133}$$

- The right-hand-side involves the pair correlation function and is called the *collision term*. It is expected to be the less important the smaller the inverse plasma parameter N_D^{-1} is. Neglecting it leads to the VLASOV *equation*

$$\left[\partial_t + L(1) - \int \mathrm{d}2\, V(1,2)F(2)\right] F(1) = 0. \tag{134}$$

- Writing L and V explicitly, we obtain the VLASOV equation

$$\boxed{\left[\partial_t + \mathbf{v}\cdot\nabla_\mathbf{r} + \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_\mathbf{v}\right] F(\mathbf{r},\mathbf{v},t) = 0} \tag{135}$$

with

$$\overline{\mathbf{B}} = \mathbf{B}_0 \tag{136}$$

and

$$\overline{\mathbf{E}} = \mathbf{E}_0 - \frac{nq}{4\pi\epsilon_0}\nabla_\mathbf{r}\int \mathrm{d}^3r' \int \mathrm{d}^3v'\, \frac{F(\mathbf{r}',\mathbf{v}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{137}$$

- We see that the *mean electric field* $\overline{\mathbf{E}}$ consists of the external part $\mathbf{E}_0$ and a HARTREE mean field due to the other particles. Indeed, because

$$nq \int \mathrm{d}^3 v\, F(\mathbf{r}, \mathbf{v}, t) = \rho(\mathbf{r}, t) \tag{138}$$

is the charge density, we may write as well

$$\overline{\mathbf{E}} = \mathbf{E}_0 - \frac{1}{4\pi\epsilon_0} \nabla_{\mathbf{r}} \int \mathrm{d}^3 r' \, \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}. \tag{139}$$

- Because we neglected magnetic effects only the external magnetic field contributes to $\overline{\mathbf{B}}$.

- It is important to keep in mind that $\overline{\mathbf{B}}$ and $\overline{\mathbf{E}}$ are mean fields. In the VLASOV equation, we neglect the part due to correlations governed by $G(1,2)$, which, by definition, we say corresponds to "collisions". It is formally, mathematically clear what we do here. However, it is not always intuitively clear to which extent a particular physical effect is included in a VLASOV treatment or not, especially far from equilibrium.

- So far we considered only a one-component plasma. If we have different species σ in the plasma the set of VLASOV equations read

$$\left[\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{q_\sigma}{m_\sigma} \left(\overline{\mathbf{E}} + \mathbf{v} \times \overline{\mathbf{B}}\right) \cdot \nabla_{\mathbf{v}}\right] F_\sigma(\mathbf{r}, \mathbf{v}, t) = 0, \tag{140}$$

with

$$\overline{\mathbf{E}} = \mathbf{E}_0 - \sum_\sigma \frac{1}{4\pi\epsilon_0} \nabla_{\mathbf{r}} \int \mathrm{d}^3 r' \, \frac{\rho_\sigma(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}. \tag{141}$$

- We can include magnetic field effects by considering the full system of VLASOV-MAXWELL equations

$$0 = \left[\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{q_\sigma}{m_\sigma} \left(\overline{\mathbf{E}} + \mathbf{v} \times \overline{\mathbf{B}}\right) \cdot \nabla_{\mathbf{v}}\right] F_\sigma(\mathbf{r}, \mathbf{v}, t), \tag{142}$$

$$\epsilon_0 \nabla \cdot \overline{\mathbf{E}} = \rho = \rho_0 + \sum_\sigma \rho_\sigma, \tag{143}$$

$$\rho_\sigma(\mathbf{r}, t) = q_\sigma n_\sigma \int \mathrm{d}^3 v\, F_\sigma(\mathbf{r}, \mathbf{v}, t), \qquad n_\sigma = \frac{N_\sigma}{V}, \tag{144}$$

$$\nabla \times \overline{\mathbf{B}} = \mu_0 \left(\mathbf{j}_0 + \sum_\sigma \mathbf{j}_\sigma\right) + \mu_0 \epsilon_0 \partial_t \overline{\mathbf{E}}, \tag{145}$$

$$\mathbf{j}_\sigma(\mathbf{r},t) = q_\sigma n_\sigma \int \mathrm{d}^3v\, \mathbf{v} F_\sigma(\mathbf{r},\mathbf{v},t), \tag{146}$$

$$\nabla \times \overline{\mathbf{E}} = -\partial_t \overline{\mathbf{B}}, \tag{147}$$

$$\nabla \cdot \overline{\mathbf{B}} = \mathbf{0}, \tag{148}$$

where ρ_0, $\mathbf{j}_0$ are external charge and current densities generating the external fields $\mathbf{E}_0, \mathbf{B}_0$. Since the latter are given explicitly anyway one just needs to solve

$$\epsilon_0 \nabla \cdot \mathbf{e} = \sum_\sigma \rho_\sigma, \qquad \nabla \cdot \mathbf{b} = 0,$$

$$\nabla \times \mathbf{b} = \mu_0 \sum_\sigma \mathbf{j}_\sigma + \mu_0 \epsilon_0 \partial_t \mathbf{e}, \qquad \nabla \times \mathbf{e} = -\partial_t \mathbf{b}$$

together with the Vlasov equation (142) self-consistently.

- The Vlasov equation (142) advances the single-particle distribution functions $F_\sigma(\mathbf{r},\mathbf{v},t)$ according to the fields $\overline{\mathbf{E}}, \overline{\mathbf{B}}$, and the Maxwell equations advance the fields according to the charge densities and the current densities $\rho_\sigma, \mathbf{j}_\sigma$, which are calculated from $F_\sigma(\mathbf{r},\mathbf{v},t)$.[4]

- The Vlasov equation (142) for a particular particle species σ looks formally like the Liouville equation for a *single* particle under the influence of a force $\overline{\mathbf{F}} = q_\sigma \left(\overline{\mathbf{E}} + \mathbf{v} \times \overline{\mathbf{B}}\right)$. The difference is that in the Vlasov equation the fields—and thus the force—depend on F_σ via eqs. (143)–(146).

E What needs to be changed in the set of equations (142)–(148) to render them relativistically correct?

- The set of equations (142)–(148) sets the stage for many of the state-of-the-art numerical plasma calculations, e.g. *"particle-in-cell"* simulations. If collisions and quantum effects are not important, the Vlasov-Maxwell system captures all the essential physics.

3.2.1 *Properties of the* Vlasov *equation*

E Because

$$N_\sigma = n_\sigma \int \mathrm{d}^3r \mathrm{d}^3v\, F_\sigma(\mathbf{r},\mathbf{v},t) = \mathrm{const} \tag{149}$$

[4] Maxwell equations (143) and (148) are actually not needed for the *propagation* of the fields but they need to be satisfied by the initial conditions.

is the number of particles of type σ, which we assume to be constant,[5] we have

$$\partial_t \int \mathrm{d}^3 r \mathrm{d}^3 v \, F_\sigma(\mathbf{r}, \mathbf{v}, t) = 0. \tag{150}$$

Show this directly from (142).

- If[6] $F(\mathbf{r}, \mathbf{v}, t = 0) > 0$ then $F(\mathbf{r}, \mathbf{v}, t) > 0$ at all times. This can be understood by assuming the opposite. Let the distribution function F become negative at $t = t_0$ and phase-space point $\mathbf{r}_0, \mathbf{v}_0$, which implies $\partial_t F|_{t_0} < 0$, $\nabla_{\mathbf{r}} F|_{\mathbf{r}_0,\mathbf{v}_0,t_0} = \nabla_{\mathbf{v}} F|_{\mathbf{r}_0,\mathbf{v}_0,t_0} = 0$. However, this is not possible according (142).

- We know that *with collisions* the classical equilibrium distribution function is a Maxwell-Boltzmann distribution function. The Vlasov equation has many more equilibrium solutions. In equilibrium, $\partial_t F = 0$ so that

$$\left[\mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{q}{m} \left(\overline{\mathbf{E}} + \mathbf{v} \times \overline{\mathbf{B}}\right) \cdot \nabla_{\mathbf{v}}\right] F(\mathbf{r}, \mathbf{v}) = 0. \tag{151}$$

Let the functions $c_i(\mathbf{r}, \mathbf{v})$, $i = 1, 2, \ldots$ be constants of the motion, i.e.,

$$\frac{\mathrm{d}c_i}{\mathrm{d}t} = \mathbf{v} \cdot \nabla_{\mathbf{r}} c_i + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} c_i = 0 \tag{152}$$

with $\dot{\mathbf{v}} = \frac{q}{m} \left(\overline{\mathbf{E}} + \mathbf{v} \times \overline{\mathbf{B}}\right)$. Then any function $F[\{c_i(\mathbf{r}, \mathbf{v})\}]$ fulfills (151), i.e.,

$$\left[\mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{q}{m} \left(\overline{\mathbf{E}} + \mathbf{v} \times \overline{\mathbf{B}}\right) \cdot \nabla_{\mathbf{v}}\right] F[\{c_i(\mathbf{r}, \mathbf{v})\}] = 0. \tag{153}$$

E Show (153).

- In the field-free case $\overline{\mathbf{E}} = \overline{\mathbf{B}} = \mathbf{0}$, where the constants of the motion are energy and momentum,

$$E = \frac{1}{2} m v^2, \qquad \mathbf{p} = m\mathbf{v}, \tag{154}$$

e.g., the Maxwell-Boltzmann distribution function

$$F(v) = \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \mathrm{e}^{-mv^2/(2k_{\mathrm{B}}T)} \tag{155}$$

---

[5] No chemical or nuclear reactions, no particle sources or sinks, no creation or annihilation of particles.

[6] We suppress the index σ if it is clear that the respective equation holds for each particle species.

is a solution of (151) but also any other function $F(v_x, v_y, v_z)$, e.g., a distribution function

$$F(\mathbf{v}) = v_0\,\delta(v_x)\,\delta(v_y)\,\delta(v_z^2 - v_0^2) \tag{156}$$

describing two counter-propagating cold beams.

E Why does it describe two counter-propagating cold beams?

- The MAXWELL-BOLTZMANN distribution function (155) is a "true" equilibrium distribution function, fulfilling also stationary[7] kinetic equations *with* a collision term on the right hand side of (133) (e.g., the BOLTZMANN equation, see below).

- Instead, the distribution (156) is an equilibrium solution of the VLASOV equation but collisions would cause the distribution to assume a MAXWELL-BOLTZMANN form on a time scale on the order of $\tau_{\text{collision}} = l/\langle v\rangle = 1/\nu$, where l is a mean free path, $\langle v\rangle$ is a mean velocity, and ν is a collision frequency, to be derived lateron.

- Hence, we expect a VLASOV description to be meaningful on *collective time scales*
$$\tau_{\text{collective}} = \frac{\lambda_{\text{D}}}{\langle v\rangle} \ll \tau_{\text{collision}}. \tag{157}$$

- The collective time scale $\tau_{\text{collective}}$ is the time scale on which the correlation function $G(1,2)$ above decays, the collisional time scale $\tau_{\text{collision}}$ is the time scale over which $F(1)$ itself would decay (towards MAXWELL-BOLTZMANN form) but using the VLASOV equation, it can't.

E What are the constants of the motion for $\overline{\mathbf{E}} = \mathbf{0}$, $\overline{\mathbf{B}} = B_0\mathbf{e}_z$? Show that

$$F(v^2, v_y + qB_0x/m, v_x - qB_0y/m)$$

is a solution of the stationary VLASOV equation in this case.

- Given a function $\Phi(F_\sigma)$, where F_σ fulfills the VLASOV equation, and

$$H(t) = \int \mathrm{d}^3r\mathrm{d}^3v\,\Phi(F_\sigma),$$

[7] That is, $\partial_t F = 0$.

it follows that H is conserved,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0. \tag{158}$$

Proof. We have[8] (dropping the σ)

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \int \mathrm{d}^3r\mathrm{d}^3v\,\partial_t\Phi(F) = \int \mathrm{d}^3r\mathrm{d}^3v\,\Phi'(F)\partial_t F \\
&= -\int \mathrm{d}^3r\mathrm{d}^3v\,\Phi'(F)\left[\mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_{\mathbf{v}}\right]F \\
&= -\int \mathrm{d}^3r\mathrm{d}^3v\,\left[\mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_{\mathbf{v}}\right]\Phi(F) \\
&= -\int \mathrm{d}^3r\mathrm{d}^3v\,\begin{pmatrix}\mathbf{v}\\ \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\end{pmatrix}\cdot\begin{pmatrix}\nabla_{\mathbf{r}}\\ \nabla_{\mathbf{v}}\end{pmatrix}\Phi(F) \\
&= -\int \underbrace{\mathrm{d}^3r\mathrm{d}^3v}_{\mathrm{d}^6X}\,\underbrace{\begin{pmatrix}\nabla_{\mathbf{r}}\\ \nabla_{\mathbf{v}}\end{pmatrix}}_{\nabla_X}\cdot\underbrace{\left[\begin{pmatrix}\mathbf{v}\\ \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\end{pmatrix}\Phi(F)\right]}_{\mathbf{J}}.
\end{aligned}$$

In the last step we used $\nabla_{\mathbf{v}}\cdot(\mathbf{v}\times\overline{\mathbf{B}}) = 0$, introduced a 6D nabla operator ∇_X, and defined a 6D phase-space current $\mathbf{J}$. Hence

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int \mathrm{d}^6X\,\nabla_X\cdot\mathbf{J} = \int \mathrm{d}\mathbf{n}\cdot\mathbf{J} = 0. \tag{159}$$

E Why does this proof also work for infinite, homogeneous systems where F does not vanish for $r\to\infty$?

- Consider the entropy

$$S(t) = -\sum_{\sigma}\int \mathrm{d}^3r\mathrm{d}^3v\,F_\sigma(\mathbf{r},\mathbf{v},t)\ln F_\sigma(\mathbf{r},\mathbf{v},t).$$

Because of (158) we have

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0, \tag{160}$$

[8] Note that for a function $g(\mathbf{r},\mathbf{v},t)$ “living” in 6D phase space

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} g(\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3r\mathrm{d}^3v = \int_{\Omega(t)}\partial_t g(\mathbf{r},\mathbf{v},t)\,\mathrm{d}^3r\mathrm{d}^3v + \int_{\partial\Omega(t)} g(\mathbf{r},\mathbf{v},t)\begin{pmatrix}\dot{\mathbf{r}}_\Omega\\ \dot{\mathbf{v}}_\Omega\end{pmatrix}\cdot\mathrm{d}\mathbf{n}$$

where $(\dot{\mathbf{r}}_\Omega, \dot{\mathbf{v}}_\Omega)$ is the 6D surface velocity of Ω and $\mathbf{n}$ is the corresponding 6D surface normal vector. This is the “LEIBNIZ integral rule” applied to our case in which, moreover, the surface does not move and thus the second integral vanishes.

i.e., the entropy is conserved in a VLASOV system, which is not surprising because there are no collisions which could bring the system to, e.g., thermal equilibrium while maximizing the entropy.

- If there are different time scales in the VLASOV dynamics such that

$$F(t) = F_{\text{slow}}(t) + F_{\text{fast}}(t)$$

one may consider the fast fluctuations as "disorder", in analogy to the disorder generated by collisions. A change in the disorder can then be measured by the entropy

$$S^* = -\int \mathrm{d}^3r\mathrm{d}^3v\, F_{\text{slow}}(\mathbf{r},\mathbf{v},t) \ln F_{\text{slow}}(\mathbf{r},\mathbf{v},t).$$

E Check that S^* indeed increases upon probability transfer from F_{slow} to F_{fast}.

- Let $\{F_{\text{stationary}}\}$ be the set of distribution functions fulfilling the stationary VLASOV equation for a particular physical setup (i.e., for given external fields). As we discussed above, this set is in general bigger than in the corresponding case with collisions. Hence, there are stationary solutions with smaller entropy $S(F_{\text{stationary}})$ than the true, maximum equilibrium entropy $S(F^{(\text{true})}_{\text{stationary}})$. We call these other solutions *metaequilibria*.

- Some of these *metaequilibria* are unstable, i.e., if one adds an infinitesimal perturbation to an $F_0 \in \{F_{\text{stationary}}\}$,

$$F(t) = F_0 + \delta F(t)$$

it may happen that $\delta F(t)$ grows in time.

- It can be shown that an equilibrium is stable if the kinetic energy is constant,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3r\mathrm{d}^3v\, \frac{1}{2}mv^2 nF = 0. \tag{161}$$

If we are able to express v^2 as a single-valued function of F,

$$v^2 = v^2(F)$$

we are back to the case (158), and (161) is certainly fulfilled. The single-valuedness can be ensured by

$$\frac{\partial F}{\partial v^2} < 0.$$

Hence, distributions $F(v^2)$ that monotonically decrease with increasing v^2 are stable VLASOV equilibria.

E Why don't we consider the other mathematical option $\partial F/\partial v^2 > 0$?

3.3 VLASOV THEORY OF PLASMA WAVES

- The full VLASOV-MAXWELL system (142)–(148) is *nonlinear* because the fields in (142) depend on $\{F_\sigma\}$. Nonlinear partial differential equations are almost always impossible to solve analytically (and numerically hard as well).
- In order to proceed analytically we *linearize* the VLASOV-MAXWELL system. We consider (for each species σ) a perturbation $\varepsilon F^{(1)}$ "on top" of an equilibrium distribution $F^{(0)}$,

$$F(\mathbf{r},\mathbf{v},t) = F^{(0)}(\mathbf{r},\mathbf{v},t) + \varepsilon F^{(1)}(\mathbf{r},\mathbf{v},t). \tag{162}$$

- Similarly for the fields

$$\overline{\mathbf{E}}(\mathbf{r},t) = \overline{\mathbf{E}}^{(0)}(\mathbf{r},t) + \varepsilon\overline{\mathbf{E}}^{(1)}(\mathbf{r},t), \quad \overline{\mathbf{B}}(\mathbf{r},t) = \overline{\mathbf{B}}^{(0)}(\mathbf{r},t) + \varepsilon\overline{\mathbf{B}}^{(1)}(\mathbf{r},t). \tag{163}$$

- In order ε^0 we have

$$\begin{aligned}
0 &= \left[\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{q_\sigma}{m_\sigma}\left(\overline{\mathbf{E}}^{(0)} + \mathbf{v}\times\overline{\mathbf{B}}^{(0)}\right)\cdot\nabla_{\mathbf{v}}\right] F_\sigma^{(0)},\\
\epsilon_0\nabla\cdot\overline{\mathbf{E}}^{(0)} &= \rho_0 + \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, F_\sigma^{(0)},\\
\nabla\times\overline{\mathbf{B}}^{(0)} &= \mu_0\left(\mathbf{j}_0 + \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, \mathbf{v}F_\sigma^{(0)}\right) + \mu_0\epsilon_0\partial_t\overline{\mathbf{E}}^{(0)},\\
\nabla\times\overline{\mathbf{E}}^{(0)} &= -\partial_t\overline{\mathbf{B}}^{(0)},\\
\nabla\cdot\overline{\mathbf{B}}^{(0)} &= \mathbf{0}.
\end{aligned}$$

- In order ε^1:

$$\begin{aligned}
&\left[\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{q_\sigma}{m_\sigma}\left(\overline{\mathbf{E}}^{(0)} + \mathbf{v}\times\overline{\mathbf{B}}^{(0)}\right)\cdot\nabla_{\mathbf{v}}\right] F_\sigma^{(1)}\\
&\qquad = -\frac{q_\sigma}{m_\sigma}\left(\overline{\mathbf{E}}^{(1)} + \mathbf{v}\times\overline{\mathbf{B}}^{(1)}\right)\cdot\nabla_{\mathbf{v}} F_\sigma^{(0)}, &(164)\\
\epsilon_0\nabla\cdot\overline{\mathbf{E}}^{(1)} &= \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, F_\sigma^{(1)}, &(165)\\
\nabla\times\overline{\mathbf{B}}^{(1)} &= \mu_0\sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, \mathbf{v}F_\sigma^{(1)} + \mu_0\epsilon_0\partial_t\overline{\mathbf{E}}^{(1)}, &(166)\\
\nabla\times\overline{\mathbf{E}}^{(1)} &= -\partial_t\overline{\mathbf{B}}^{(1)}, &(167)\\
\nabla\cdot\overline{\mathbf{B}}^{(1)} &= \mathbf{0}. &(168)
\end{aligned}$$

- We expect that this set of equations describes *perturbations* of a plasma around an equilibrium $\{F_\sigma^{(0)}\}$, such as *plasma waves*.

3.3.1 *Perturbations of a field-free plasma equilibrium*

- Let us consider

$$\overline{\mathbf{E}}^{(0)} = \overline{\mathbf{B}}^{(0)} = \mathbf{0}, \quad \rho_0 = 0,\ \mathbf{j}_0 = \mathbf{0}, \qquad F_\sigma^{(0)} = F_\sigma^{(0)}(\mathbf{v}), \tag{169}$$

i.e.,

$$\sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, F_\sigma^{(0)} = 0, \tag{170}$$

$$\mu_0 \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, \mathbf{v} F_\sigma^{(0)} = \mathbf{0}. \tag{171}$$

- We wish to study the simplest case of an *electrostatic* perturbation where

$$\forall t \quad \overline{\mathbf{B}}^{(1)} = \mathbf{0} \qquad \Rightarrow \qquad \nabla \times \overline{\mathbf{E}}^{(1)} = \mathbf{0} \tag{172}$$

so that $\overline{\mathbf{E}}^{(1)}$ can be expressed as a gradient of a scalar potential,

$$\overline{\mathbf{E}}^{(1)} = -\nabla\Phi^{(1)}. \tag{173}$$

E Give explicit examples for perturbations $F^{(1)}$ that do not lead to any $\overline{\mathbf{B}}^{(1)}$.

- We then have for (164) and (165)

$$\left[\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{r}}\right] F_\sigma^{(1)} = \frac{q_\sigma}{m_\sigma}\nabla\Phi^{(1)}\cdot\nabla_{\mathbf{v}} F_\sigma^{(0)} \tag{174}$$

and

$$\epsilon_0 \nabla^2\Phi^{(1)} = -\sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3v\, F_\sigma^{(1)}, \tag{175}$$

respectively.

- Linear partial differential equations can be tackled using the Fourier-Laplace transform,

$$g_{\mathbf{k}}(\mathbf{v},p) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-pt} \underbrace{\frac{1}{(2\pi)^3}\int \mathrm{d}^3r\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} g(\mathbf{r},\mathbf{v},t),}_{\text{Fourier-transform } g_{\mathbf{k}}(\mathbf{v},t)} \qquad \mathrm{Re}(p) \geq p_0 \tag{176}$$

$$= \frac{1}{(2\pi)^3}\int d^3r\ e^{-i\mathbf{k}\cdot\mathbf{r}} \underbrace{\int_0^\infty dt\ e^{-pt}\, g(\mathbf{r},\mathbf{v},t)}_{\text{Laplace}-\text{transform } g(\mathbf{r},\mathbf{v},p)},$$

where p_0 has to be chosen such that $\int_0^\infty dt\ e^{-pt}\, g(\mathbf{r},\mathbf{v},t)$ converges.

The inverse transform reads

$$g(\mathbf{r},\mathbf{v},t) = \int d^3k\ e^{i\mathbf{k}\cdot\mathbf{r}} \int_{p_0-i\infty}^{p_0+i\infty} \frac{dp}{2\pi i}\, e^{pt}\, g_{\mathbf{k}}(\mathbf{v},p), \tag{177}$$

where the integration contour p in the complex p-plane has to be to the right of all poles $\{p_j\}$ of $g_{\mathbf{k}}(\mathbf{v},p)$.

- Transforming (174) yields for the left hand side

$$\begin{aligned}
&\int_0^\infty dt\ e^{-pt}\frac{1}{(2\pi)^3}\int d^3r\ e^{-i\mathbf{k}\cdot\mathbf{r}}\ [\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{r}}]\, F_\sigma^{(1)}(\mathbf{r},\mathbf{v},t)\\
&= \int_0^\infty dt\ e^{-pt}\partial_t F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},t)\\
&\quad + \frac{1}{(2\pi)^3}\int d^3r\ e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathbf{v}\cdot\nabla_{\mathbf{r}} F_\sigma^{(1)}(\mathbf{r},\mathbf{v},p)\\
&= \int_0^\infty dt\,\partial_t\left[e^{-pt}F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},t)\right] + \int_0^\infty dt\, p\, e^{-pt}F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},t)\\
&\quad + \underbrace{\frac{1}{(2\pi)^3}\int d^3r\,\mathbf{v}\cdot\nabla_{\mathbf{r}}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}\, F_\sigma^{(1)}(\mathbf{r},\mathbf{v},p)\right]}_{0} + \frac{1}{(2\pi)^3}\int d^3r\, i\mathbf{k}\cdot\mathbf{v}F_\sigma^{(1)}(\mathbf{r},\mathbf{v},p)\ e^{-i\mathbf{k}\cdot\mathbf{r}}\\
&= -F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},t=0) + (p + i\mathbf{k}\cdot\mathbf{v})F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},p).
\end{aligned}$$

- On the right hand side of (174) $F_\sigma^{(0)}$ depends neither on $\mathbf{r}$ nor t. Hence

$$\int_0^\infty dt\ e^{-pt}\frac{1}{(2\pi)^3}\int d^3r\ e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{q_\sigma}{m_\sigma}\nabla\Phi^{(1)}\cdot\nabla_{\mathbf{v}}F_\sigma^{(0)} = \frac{q_\sigma}{m_\sigma}[i\mathbf{k}\cdot\nabla_{\mathbf{v}}F_\sigma^{(0)}(\mathbf{v})]\Phi_{\mathbf{k}}^{(1)}(p)$$

and thus

$$(p + i\mathbf{k}\cdot\mathbf{v})F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},p) = F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},t=0) + \frac{q_\sigma}{m_\sigma}[i\mathbf{k}\cdot\nabla_{\mathbf{v}}F_\sigma^{(0)}(\mathbf{v})]\Phi_{\mathbf{k}}^{(1)}(p). \tag{178}$$

- For (175) we have

$$\epsilon_0 k^2\Phi_{\mathbf{k}}^{(1)}(p) = \sum_\sigma q_\sigma n_\sigma\int d^3v\, F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v},p). \tag{179}$$

E Show (179).

- We have switched from $(\mathbf{r}, t)$-space to $(\mathbf{k}, p)$-space [or $(\mathbf{k}, \omega)$ with $\omega = \mathrm{i}p$, see below]. In doing so, we successfully converted a set of partial differential equations into a set of algebraic equations.

- Eqs. (178) and (179) can be solved for $\Phi_{\mathbf{k}}^{(1)}(p)$ by eliminating $F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, p)$:

$$\epsilon_0 k^2 \Phi_{\mathbf{k}}^{(1)}(p) \underbrace{\left(1 - \sum_\sigma \frac{q_\sigma^2 n_\sigma}{m_\sigma \epsilon_0 k^2} \int \mathrm{d}^3 v \, \frac{\mathrm{i}\mathbf{k} \cdot \nabla_{\mathbf{v}} F_\sigma^{(0)}(\mathbf{v})}{p + \mathrm{i}\mathbf{k} \cdot \mathbf{v}}\right)}_{D} = \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3 v \, \frac{F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, t=0)}{p + \mathrm{i}\mathbf{k} \cdot \mathbf{v}}.$$

- The quantity D is called the *dielectric function*,

$$D(\mathbf{k}, \omega) = 1 + \sum_\sigma \frac{q_\sigma^2 n_\sigma}{\epsilon_0 m_\sigma k^2} \int \mathrm{d}^3 v \, \frac{\mathbf{k} \cdot \nabla_{\mathbf{v}} F_\sigma^{(0)}(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}}, \qquad \omega = \mathrm{i}p, \ \mathrm{Re}(p) \geq p_0. \tag{180}$$

  In fact, it emerges as a factor together with ϵ_0, like the relative dielectric permittivity ϵ in the MAXWELL equations for continuous media.

- Introducing

$$v_\parallel = \frac{\mathbf{k} \cdot \mathbf{v}}{k}, \qquad k = |\mathbf{k}|, \qquad \mathbf{v}_\perp = \mathbf{v} - v_\parallel \frac{\mathbf{k}}{k} \tag{181}$$

  we can write (180) as

$$D(\mathbf{k}, \omega) = 1 + \sum_\sigma \frac{q_\sigma^2 n_\sigma}{\epsilon_0 m_\sigma k^2} \int \mathrm{d}^2 v_\perp \int \mathrm{d}v_\parallel \, \frac{\partial_{v_\parallel} F_\sigma^{(0)}(\mathbf{v})}{\omega/k - v_\parallel}. \tag{182}$$

- Defining

$$F_{\parallel\sigma}^{(0)}(v_\parallel) = \int \mathrm{d}^2 v_\perp \, F_\sigma^{(0)}(\mathbf{v}) = \int \mathrm{d}^3 v \, F_\sigma^{(0)}(\mathbf{v}) \, \delta\left(v_\parallel - \frac{\mathbf{k} \cdot \mathbf{v}}{k}\right), \tag{183}$$

  we can write

$$D(\mathbf{k}, \mathrm{i}p) = 1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \int \mathrm{d}v_\parallel \, \frac{\partial_{v_\parallel} F_{\parallel\sigma}^{(0)}(v_\parallel)}{v_\parallel - \mathrm{i}p/k}, \tag{184}$$

  $\mathrm{Re}(p) \geq p_0$, where

$$\omega_{\mathrm{p}\sigma} = \left(\frac{q_\sigma^2 n_\sigma}{\epsilon_0 m_\sigma}\right)^{1/2} \tag{185}$$

  is the plasma frequency of the species σ [cf. (24)].

- The potential reads

$$\Phi_{\mathbf{k}}^{(1)}(p) = \frac{-\mathrm{i}}{\epsilon_0 k^3 D(\mathbf{k}, \mathrm{i}p)} \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}v_\parallel \, \frac{F_{\parallel\sigma\mathbf{k}}^{(1)}(v_\parallel, t=0)}{v_\parallel - \mathrm{i}p/k} \tag{186}$$

where, in analogy with (183),

$$F_{\parallel\sigma\mathbf{k}}^{(1)}(v_\parallel, t=0) = \int \mathrm{d}^3v \, F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, t=0) \, \delta\left(v_\parallel - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right). \tag{187}$$

- We are interested in how the *modes* $\mathbf{k}$ of the potential evolve in time. To that end we take the inverse LAPLACE transform and obtain

$$\Phi_{\mathbf{k}}^{(1)}(t) = \int_{p_0 - \mathrm{i}\infty}^{p_0 + \mathrm{i}\infty} \frac{\mathrm{d}p}{2\pi\mathrm{i}} \, \mathrm{e}^{pt} \, \Phi_{\mathbf{k}}^{(1)}(p), \tag{188}$$

or

$$k^2 \Phi_{\mathbf{k}}^{(1)}(t) = \int_{p_0 - \mathrm{i}\infty}^{p_0 + \mathrm{i}\infty} \frac{\mathrm{d}p}{2\pi\mathrm{i}} \, \frac{\mathrm{e}^{pt}}{\epsilon_0 D(\mathbf{k}, \mathrm{i}p)} \sum_\sigma q_\sigma n_\sigma \int \mathrm{d}v_\parallel \, \frac{F_{\parallel\sigma\mathbf{k}}^{(1)}(v_\parallel, t=0)}{\mathrm{i}k v_\parallel + p}. \tag{189}$$

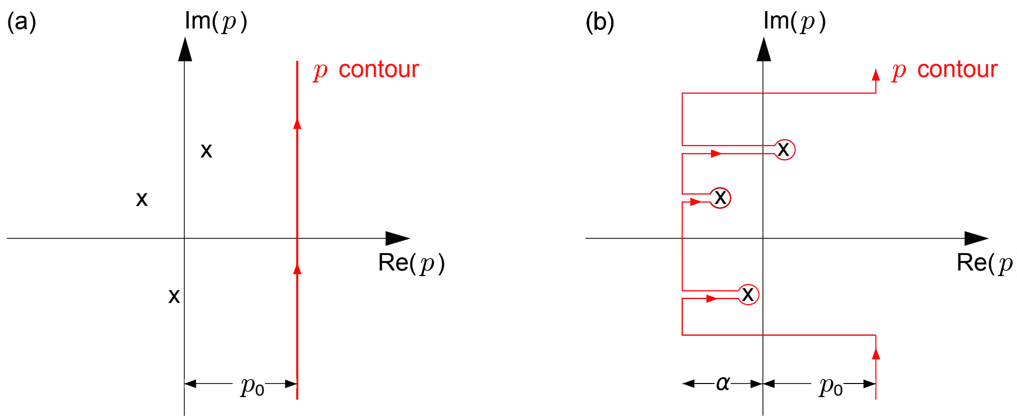
- In a few cases, this expression can be evaluated in a straightforward way.

E Calculate $\Phi_{\mathbf{k}}^{(1)}(t)$ for $\sigma = \mathrm{e}, \mathrm{i}$ (electrons and ions), $F_\mathrm{e}^{(0)} = F_\mathrm{i}^{(0)} = \delta(\mathbf{v})$ (cold plasma), $F_\mathrm{e}^{(1)} = \hat{F}_\mathrm{e}^{(1)} \sin \kappa x$, and $F_\mathrm{i}^{(1)} = 0$.

- In general, however, the evaluation of (189) is quite involved. The integration contour in the complex p-plane must be chosen such that all poles of $\Phi_{\mathbf{k}}^{(1)}(p)$ are to the left. The poles $\{p_j(k)\}$ are determined by

$$D[\mathbf{k}, \mathrm{i}p_j(k)] = 0. \tag{190}$$

- We wish to deform the integration contour as depicted in the following figure from (a) to (b):



- As $\Phi_{\mathbf{k}}^{(1)}(p)$ is only defined for $\mathrm{Re}(p) \geq p_0$ we need an analytic continuation of $\Phi_{\mathbf{k}}^{(1)}(p)$ for $\mathrm{Re}(p) < p_0$, which turns out to be just of the same functional form as (186) with the same dielectric function (184) *if* the velocity integrals that appear in (186), (184) are defined in the following way:

  The velocity integrals are of the type

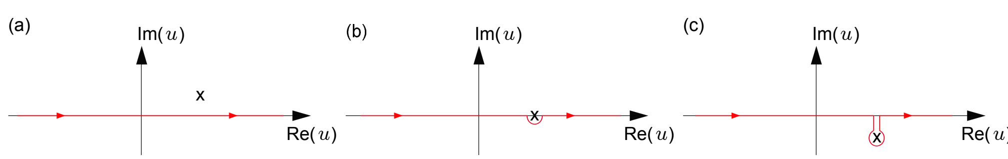
$$h(p) = \int_{-\infty}^{\infty} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k}$$

  so that in the imaginary u plane (corresponding to $v_{\parallel}$) the pole is at

$$u_1 = \frac{\mathrm{i}p}{k},$$

  i.e., above, on, or below the real u-axis.

- Assuming that $f(u)$ itself is analytic, the so-called LANDAU contours show us how to integrate in the three cases:



In formulas:

$$h(p) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k} & \mathrm{Re}\, p \geq 0, \\ \displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k} + 2\pi\mathrm{i} f(\mathrm{i}p/k) & \mathrm{Re}\, p \leq 0. \end{cases}$$

E Show that for $\operatorname{Re} p \to 0$ both expressions give

$$h(p) = \mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k} + \pi \mathrm{i} f(\mathrm{i}p/k),$$

where

$$\mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k} = \lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{\mathrm{i}p/k-\epsilon} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k} + \int_{\mathrm{i}p/k+\epsilon}^{\infty} \frac{\mathrm{d}u\, f(u)}{u - \mathrm{i}p/k} \right)$$

is the CAUCHY principal value.

- Now, having properly defined the velocity integrals, we come back to the integration path (b) in the complex p plane depicted above. We see that (188) becomes

$$\Phi_{\mathbf{k}}^{(1)}(t) = \sum_j \mathrm{R}_j \, \mathrm{e}^{p_j(k)t} + \left(\int_{-\mathrm{i}\infty+p_0}^{-\mathrm{i}\infty-\alpha} + \int_{-\alpha-\mathrm{i}\infty}^{-\alpha+\mathrm{i}\infty} + \int_{\mathrm{i}\infty-\alpha}^{\mathrm{i}\infty+p_0} \right) \Phi_{\mathbf{k}}^{(1)}(p) \, \mathrm{e}^{pt} \frac{\mathrm{d}p}{2\pi\mathrm{i}},$$

where

$$\mathrm{R}_j = \lim_{p \to p_j} (p - p_j)\, \Phi_{\mathbf{k}}^{(1)}(p)$$

are the residues.

- The integrals $\int_{-\mathrm{i}\infty+p_0}^{-\mathrm{i}\infty-\alpha}$ and $\int_{\mathrm{i}\infty-\alpha}^{\mathrm{i}\infty+p_0}$ vanish if $\lim_{|p|\to\infty} \Phi_{\mathbf{k}}^{(1)}(p) = 0$.

- What about the $\int_{-\alpha-\mathrm{i}\infty}^{-\alpha+\mathrm{i}\infty}$ integral? If $|\alpha|$ is chosen sufficiently large, the exponential factor

$$\mathrm{e}^{\alpha t} = \mathrm{e}^{-|\alpha| t}$$

will, for $t \to \infty$, suppress contributions from this part of the contour. By neglecting these contour parts, we neglect *transient* effects in the plasma due to the sudden disturbance of F at $t = 0$. What we then obtain are the so-called *time-asymptotic solutions* for $\Phi_{\mathbf{k}}^{(1)}(t)$.

- Hence, we have the result

$$\Phi_{\mathbf{k}}^{(1)}(t \to \infty) = \sum_j \mathrm{R}_j \, \mathrm{e}^{p_j(k)t} = \sum_j \mathrm{R}_j \, \mathrm{e}^{-\mathrm{i}\omega_j t}, \qquad \omega_j = \mathrm{i}p_j. \tag{191}$$

The frequencies ω_j are the poles of the dielectric function [see (190)],

$$D(\mathbf{k}, \omega_j) = 0. \tag{192}$$

- The equation (192) gives the frequency of *plasma waves* as a function of the wave number, i.e., it is the *dispersion relation*. In other words, (192) yields the *eigenmodes* of a plasma.

E Recapitulate all the input required to tell the dispersion relation.

3.3.2 *Electrostatic (Langmuir) waves and Landau damping*

- Equation (192) can be written (dropping the subscript j)

$$D(\mathbf{k},\omega) = 1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \int \mathrm{d}u \, \frac{\partial_u F^{(0)}_{\parallel\sigma}(u)}{u - \omega/k} = 0. \tag{193}$$

- The roots $\omega = \omega_r + \mathrm{i}\omega_i$ are in general complex, and often $\omega_i = \mathrm{Im}\,\omega$ is much smaller than $\omega_r = \mathrm{Re}\,\omega$. If it was not, the wave would be damped as fast as the transient dynamics dies out. Since we already specialized in asymptotic, large-time solutions we cannot treat rapidly damped waves by the method dicussed here.

- We expand around $\omega_i = 0$, giving

$$1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \left(1 + \mathrm{i}\omega_i \frac{\partial}{\partial\omega_r}\right) \lim_{\varepsilon\to 0^+} \int \mathrm{d}u \, \frac{\partial_u F^{(0)}_{\parallel\sigma}(u)}{u - \omega_r/k - \mathrm{i}\varepsilon} = 0. \tag{194}$$

- Using the Plemelj formular

$$\lim_{\varepsilon\to 0^+} \int \mathrm{d}u \, \frac{g(u)}{u - v_\varphi - \mathrm{i}\varepsilon} = \mathrm{P}\int \mathrm{d}u \, \frac{g(u)}{u - v_\varphi} + \pi\mathrm{i} g(u = v_\varphi) \tag{195}$$

  with $g(u) = \partial_u F^{(0)}_{\parallel\sigma}(u)$ and assuming that the phase velocity $v_\varphi = \omega_r/k$ of the wave is large compared to the thermal velocity $v_{\mathrm{th}}^2 = 2k_\mathrm{B}T/m$,

$$v_{\mathrm{th}} \ll v_\varphi = \frac{\omega_r}{k},$$

  we can expand the principal value integral in u

$$\mathrm{P}\int \mathrm{d}u \, \frac{g(u)}{u - v_\varphi} = -\int \mathrm{d}u \, g(u) \left[\frac{1}{v_\varphi} + \frac{u}{v_\varphi^2} + \frac{u^2}{v_\varphi^3} + \cdots\right]. \tag{196}$$

- Let us specialize on electrons $m = m_\mathrm{e}$, neglecting the ions (whose contribution is by m/m_i smaller in the sums over σ).

- For a Maxwellian distribution[9] for the electrons

$$F^{(0)}_\parallel = \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{1/2} \mathrm{e}^{-mu^2/(2k_\mathrm{B}T)} \tag{197}$$

[9] Now in 1D because $v_\perp$ is already integrated away.

we find

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{\omega_\mathrm{p}^2}{k^2}\int_{-\infty}^{\infty} \mathrm{d}u\, \frac{\partial_u F_\parallel^{(0)}(u)}{u-\omega_r/k-\mathrm{i}\varepsilon} \\
&= \frac{\omega_\mathrm{p}^2}{\omega_r^2} + 3\frac{\omega_\mathrm{p}^4}{\omega_r^4}k^2\lambda_\mathrm{D}^2 + \cdots + \mathrm{i}\pi\frac{\omega_\mathrm{p}^2}{k^2}\,\partial_u F_\parallel^{(0)}(u)|_{u=\omega_r/k} \qquad (198)\\
&= \frac{\omega_\mathrm{p}^2}{\omega_r^2} + 3\frac{\omega_\mathrm{p}^4}{\omega_r^4}k^2\lambda_\mathrm{D}^2 + \cdots - \mathrm{i}\pi\frac{\omega_\mathrm{p}^2}{k^2}\left(\frac{m}{k_\mathrm{B}T}\right)^{3/2}\frac{1}{\sqrt{2\pi}}\frac{\omega_r}{k}\ \mathrm{e}^{-m/(2k_\mathrm{B}T)(\omega_r/k)^2}, \\
&= \frac{\omega_\mathrm{p}^2}{\omega_r^2} + 3\frac{\omega_\mathrm{p}^4}{\omega_r^4}(k\lambda_\mathrm{D})^2 + \cdots - \mathrm{i}\sqrt{\frac{\pi}{2}}\frac{\omega_r}{\omega_\mathrm{p}}\frac{1}{(k\lambda_\mathrm{D})^3}\ \exp\left[-\frac{1}{2}\left(\frac{\omega_r}{\omega_\mathrm{p}}\right)^2\frac{1}{(k\lambda_\mathrm{D})^2}\right],
\end{aligned}$$

where the Debye length (16) and

$$\omega_\mathrm{p}^2\lambda_\mathrm{D}^2 = \frac{k_\mathrm{B}T}{m} = \frac{1}{2}v_\mathrm{th}^2 \qquad (199)$$

were used. The last, imaginary term corresponds to the last term in (195). Because $k\lambda_\mathrm{D}$ is small, the $(k\lambda_\mathrm{D})^{-2}$ in the exponential ensures that this term is small.

E Show (198).

- Plugging our result into (194) yields (considering only electrons)

$$1-\left(1+\mathrm{i}\omega_i\frac{\partial}{\partial\omega_r}\right)\left(\frac{\omega_\mathrm{p}^2}{\omega_r^2}+3\frac{\omega_\mathrm{p}^4}{\omega_r^4}(k\lambda_\mathrm{D})^2+\cdots-\mathrm{i}\sqrt{\frac{\pi}{2}}\frac{\omega_r}{\omega_\mathrm{p}}\frac{1}{(k\lambda_\mathrm{D})^3}\ \exp\left[-\frac{1}{2}\left(\frac{\omega_r}{\omega_\mathrm{p}}\right)^2\frac{1}{(k\lambda_\mathrm{D})^2}\right]\right)=0.$$

We require that $\omega_i \ll \omega_r$ and $k\lambda_\mathrm{D} \ll 1$. Hence, in zeroth order, we have $1-\frac{\omega_\mathrm{p}^2}{\omega_r^2}=0$, and up to first order for the real part

$$1-\frac{\omega_\mathrm{p}^2}{\omega_r^2}-3\frac{\omega_\mathrm{p}^4}{\omega_r^4}(k\lambda_\mathrm{D})^2=0$$

so that (using the zeroth order)

$$\omega_r^2 \simeq \omega_\mathrm{p}^2\left(1+3(k\lambda_\mathrm{D})^2\right)$$

and thus

$$\boxed{\omega_r \simeq \omega_\mathrm{p}\left(1+\frac{3}{2}(k\lambda_\mathrm{D})^2\right)}\,, \qquad (200)$$

which is called the Bohm-Gross *dispersion relation*.

- For the imaginary part the leading terms yield

$$-\mathrm{i}\omega_i \frac{\partial}{\partial\omega_r}\frac{\omega_\mathrm{p}^2}{\omega_r^2} + \mathrm{i}\sqrt{\frac{\pi}{2}}\frac{\omega_r}{\omega_\mathrm{p}}\frac{1}{(k\lambda_\mathrm{D})^3}\,\exp\left[-\frac{1}{2}\left(\frac{\omega_r}{\omega_\mathrm{p}}\right)^2\frac{1}{(k\lambda_\mathrm{D})^2}\right] = 0.$$

  Hence

$$\begin{aligned}\omega_i &= -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_r^4}{\omega_\mathrm{p}^4}\frac{\omega_\mathrm{p}}{(k\lambda_\mathrm{D})^3}\,\exp\left[-\frac{1}{2}\left(\frac{\omega_r}{\omega_\mathrm{p}}\right)^2\frac{1}{(k\lambda_\mathrm{D})^2}\right]\\ &\simeq -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_\mathrm{p}}{(k\lambda_\mathrm{D})^3}\,\exp\left[-\frac{1}{2}\left(1+3(k\lambda_\mathrm{D})^2\right)\frac{1}{(k\lambda_\mathrm{D})^2}\right]\end{aligned}$$

  so that

$$\boxed{\omega_i \simeq -\sqrt{\frac{\pi}{8}}\frac{\omega_\mathrm{p}}{(k\lambda_\mathrm{D})^3}\,\exp\left[-\frac{1}{2(k\lambda_\mathrm{D})^2}-\frac{3}{2}\right]}. \qquad (201)$$

- We see that there is a *thermal correction* $\sim k^2\lambda_\mathrm{D}^2$ in the Bohm-Gross dispersion relation (200). In leading order the frequency is just the plasma frequency, $\omega_r = \omega_\mathrm{p}$, which we found already in section 1.3 where we did not consider any thermal effects.

- Oscillations fulfilling the dispersion relation (200) are called Langmuir *oscillations*.

- We further find that for $k_\mathrm{B}T > 0 \Rightarrow \lambda_\mathrm{D} > 0$, i.e., a *warm plasma*, the Langmuir *oscillations* with finite k propagate. Hence, we have electrostatic (or Langmuir) *waves*.

- Moreover, eq. (201) tells us that these oscillations and waves are damped. In fact, the Fourier-backtransformed potential,

$$\Phi^{(1)}(\mathbf{r},t) \sim \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_r t)}\;\mathrm{e}^{\omega_i t},$$

  shows that the waves are damped for $\omega_i < 0$ according (201).

- It is striking that damping occurs without particle collisions. This *collisionless damping* is called Landau *damping*. It is a quite universal phenomenon which appears in various disguises in all branches of physics.

- In (198) we see that the imaginary part ω_i arises from the derivative of the zeroth order distribution function, evaluated at the phase velocity $v_\varphi = \omega_r/k$ of the wave,

$$\omega_i \sim \partial_u F_\parallel^{(0)}(u)|_{u=\omega_r/k}.$$

Therefore LANDAU damping is a *resonant phenomenon*. The fact that the derivative of the distribution function shows up indicates that the damping occurs because there are more particles that are slower than v_φ than particles that are faster (for distributions $\partial_u F_\parallel^{(0)}(u) < 0$). Hence, on average, the wave looses energy.

- We assumed for our derivation that $k\lambda_\mathrm{D} \ll 1$ (i.e., long wavelengths) and $\omega_i \ll \omega_r$ (i.e., weak damping).

E Check that our results (200), (201) are consistent with these conditions.

E Plot the LANDAU damping time ω_i^{-1} in units of $2\pi/\omega_\mathrm{p}$ vs $k\lambda_\mathrm{D}$. In which part of the plot is our theory valid?

- Clearly, the long wavelengths

$$\lambda_\mathrm{Langmuir} = \frac{2\pi}{k} \tag{202}$$

have to "fit into the plasma", i.e., given the size of the plasma L, we must have that $L \gg k^{-1} \gg \lambda_\mathrm{D}$.

Damping without entropy increase?

- We obtained an imaginary part of the frequency, leading to the damping of the perturbation *in the field* (or electrostatic potential). Where does the energy go? There can't be real dissipation leading to increased temperature or entropy because there are no collisions.

- This seeming paradox is resolved by calculating what the particles do upon the perturbation. From (178) we have

$$F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, p) = \frac{F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, t=0)}{p + \mathrm{i}\mathbf{k}\cdot\mathbf{v}} + \frac{q_\sigma}{m_\sigma} \frac{\mathrm{i}\mathbf{k}\cdot\nabla_\mathbf{v} F_\sigma^{(0)}(\mathbf{v})}{p + \mathrm{i}\mathbf{k}\cdot\mathbf{v}} \Phi_\mathbf{k}^{(1)}(p).$$

- The LAPLACE inverse

$$F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, t) = \int_{p_0 - \mathrm{i}\infty}^{p_0 + \mathrm{i}\infty} \frac{\mathrm{d}p}{2\pi\mathrm{i}} F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, p)\, \mathrm{e}^{pt}, \tag{203}$$

with (186) plugged in, is carried out analogously to the inverse LAPLACE transform of $\Phi_\mathbf{k}^{(1)}(p)$ itself, extensively discussed above.

- In addition to the poles of $\Phi_{\mathbf{k}}^{(1)}(p)$ at ω_j where $D(\mathbf{k}, \omega_j) = 0$ there is also a pole in (203) at $p = -\mathrm{i}\mathbf{k} \cdot \mathbf{v}$.
- Hence, instead of (191)

$$\Phi_{\mathbf{k}}^{(1)}(t \to \infty) = \sum_j \mathrm{R}_j \ \mathrm{e}^{-\mathrm{i}\omega_j t}, \qquad \omega_j = \mathrm{i}p_j,$$

  for $F_{\sigma\mathbf{k}}^{(1)}(t \to \infty)$ we have

$$F_{\sigma\mathbf{k}}^{(1)}(t \to \infty) = \tilde{\mathrm{R}}_{\text{ball}} \ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{v}t} + \sum_j \tilde{\mathrm{R}}_j \ \mathrm{e}^{-\mathrm{i}\omega_j t}, \tag{204}$$

  where $\tilde{\mathrm{R}}_{\text{ball}}$ and $\tilde{\mathrm{R}}_j$ are some coefficients determined by the residues of the poles of $F_{\sigma\mathbf{k}}^{(1)}(\mathbf{v}, p)$.

E Calculate $\tilde{\mathrm{R}}_{\text{ball}}$ and $\tilde{\mathrm{R}}_j$.

- While the sum over j contains the contributions damped as those in the electrostatic potential, the new term $\tilde{\mathrm{R}}_{\text{ball}} \ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{v}t}$ is not damped because both $\mathbf{k}$ and $\mathbf{v}$ are real. This term is called the *ballistic term*. It "memorizes" the perturbation at $t = 0$ because there are no particle collisions that could erase the information about the perturbation.
- Although the ballistic term does not decrease with increasing time it becomes highly oscillatory in $\mathbf{v}$-space. This is the explanation why it disappears in the asymptotic-time solution of the potential. As [cf. (175)]

$$\epsilon_0 \nabla^2 \Phi^{(1)} = -\sum_\sigma q_\sigma n_\sigma \int \mathrm{d}^3 v \, F_\sigma^{(1)},$$

  the velocity integration will remove the contribution of the ballistic source term on the right hand side.
- The effect that information is lost (in a certain obervable, here the electrostatic wave) because of fast oscillations of integrands is called *phase mixing*.

3.3.3 *Ion-acoustic waves*

- So far we always neglected the ion dynamics with the argument that the ion masses are much greater than the electron masses. However, it may happen that the electrons are much warmer than the ions, $T_{\mathrm{e}} \gg T_{\mathrm{i}}$,

$$v_{\text{th,i}} < \frac{\omega}{k} = v_\varphi < v_{\text{th,e}}. \tag{205}$$

- We can then apply the expansion (196) to the ions,

$$\mathrm{P}\int \mathrm{d}u\,\frac{\partial_u F^{(0)}_{\parallel,\mathrm{i}}}{v_\varphi - u} \simeq \int \mathrm{d}u\,\partial_u F^{(0)}_{\parallel,\mathrm{i}}\left[\frac{1}{v_\varphi} + \frac{u}{v_\varphi^2} + \frac{u^2}{v_\varphi^3} + \cdots\right],$$

  but use

$$\mathrm{P}\int \mathrm{d}u\,\frac{\partial_u F^{(0)}_{\parallel,\mathrm{e}}}{v_\varphi - u} = \mathrm{P}\int \mathrm{d}u\,\frac{2u\partial_{u^2} F^{(0)}_{\parallel,\mathrm{e}}}{v_\varphi - u} \simeq -\int_{-\infty}^{\infty}\mathrm{d}u\,2\partial_{u^2}F^{(0)}_{\parallel,\mathrm{e}}$$

  for the electrons.

- Plugging this into (195) and the result into (194) yields with $m_\mathrm{e} = m$ and $m_\mathrm{i} = M$

$$\boxed{\omega_r^2 = \frac{k^2 C_s^2}{1 + k^2\lambda_\mathrm{D}^2}}, \qquad \lambda_\mathrm{D} = \sqrt{\frac{\epsilon_0 k_\mathrm{B} T_\mathrm{e}}{e^2 n_\mathrm{e}}}, \tag{206}$$

$$\omega_i = -\frac{\omega_r\sqrt{\pi/8}}{(1+k^2\lambda_\mathrm{D}^2)^{3/2}}\left[\left(\frac{T_\mathrm{e}}{T_\mathrm{i}}\right)^{3/2}\exp\left(-\frac{T_\mathrm{e}/T_\mathrm{i}}{2(1+k^2\lambda_\mathrm{D}^2)}\right) + \sqrt{\frac{m}{M}}\right] \tag{207}$$

  where

$$\boxed{C_s = \sqrt{\frac{k_\mathrm{B} T_\mathrm{e}}{M}}} \tag{208}$$

  is the *ion-acoustic sound speed*. Note that there is the electron temperature but the ion mass in this expression.

- Because of our assumption (205) these expressions are only valid if

$$\sqrt{\frac{k_\mathrm{B} T_\mathrm{i}}{M}} < \frac{\omega_r}{k} < \sqrt{\frac{k_\mathrm{B} T_\mathrm{e}}{m}}. \tag{209}$$

[E] With decreasing electron temperature the damping with the rate $\sim |\omega_i|$ increases and the sound speed decreases. Give a physical explanation for this.

[E] Derive (206)–(208).

- The ion-acoustic dispersion relation (206) is qualitatively very different from the electron plasma waves' one (200) in the long-wavelength regime $(k\lambda_\mathrm{D})^2 \ll 1$:

  - For electron plasma waves $\omega_r \simeq \omega_\mathrm{p}$, i.e., all waves oscillate with almost the same frequency;
  - For ion-acoustic waves $\omega_r \sim k$ and $\omega_r/k = v_\varphi \simeq C_s \simeq v_\mathrm{group} = \partial_k \omega_r$ is independent of k, i.e., all modes travel with the same speed (like light waves where $\omega = ck$ with c the speed of light).

- The damping due to $\omega_i < 0$ in (207) is weak because
  - the slope of the electron distribution $F^{(0)}_{\parallel,\mathrm{e}}(u)$ is rather flat at $u = v_\varphi$ while
  - only a few ions can contribute to damping because $F^{(0)}_{\parallel,\mathrm{i}}(u)$ is small at $u = v_\varphi$.

4

PLASMA AS FLUID

- The kinetic description of plasma introduced in the previous chapter can be simplified further.

- The idea is to take *moments of the kinetic equations* by integrating-away the velocity-space degrees of freedom. Instead of "watching" particles in phase space one follows "lumps of particles" in position space only.

- We are interested in macroscopic entities like density $n(\mathbf{r}, t)$, fluid velocity $\mathbf{u}(\mathbf{r}, t)$, current density $\mathbf{j}(\mathbf{r}, t)$ etc., i.e., all $\in \mathbb{R}^{3+1}$ instead of $F(\mathbf{r}, \mathbf{v}, t)$ $\in \mathbb{R}^{6+1}$.

- This simplification allows us to apply plasma theory to real-world problems of, e.g., astrophysical interest.

- Of course, this reduction of complexity is not for free. Some results obtained through a kinetic description (e.g., LANDAU damping, collision frequencies, equations of state) cannot be derived from fluid theory but enter it as input.[1]

> *"It should be clear that the fluid theory, though of practical use, relies heavily on the cunning of its user."*
>
> From KRALL & TRIVELPIECE, *Principles of Plasma Physics*

- The use of equations of state to close and to cut the system of fluid equations requires the plasma to be *collisional*. This is because many collisions must occur on the length and time scales we are interested in in order to render the assumption of a *local thermodynamic equilibrium* and an equation of state meaningful.

[1] This is the reason why we pursue a "top-down" approach where we started (in the previous chapter) with LIOUVILLE and KLIMONTOVICH.

4.1 DERIVATION OF FLUID EQUATIONS

- Let us start with the first equation of the BBGKY hierarchy (in electrostatic approximation) (133), which we write as[2]

$$\boxed{\left[\partial_t + \mathbf{v}\cdot\nabla_\mathbf{r} + \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_\mathbf{v}\right] F(\mathbf{r},\mathbf{v},t) = \left.\frac{\partial F}{\partial t}\right|_\mathrm{c}} \tag{210}$$

(Boltzmann equation) with the *collision term*

$$\left.\frac{\partial F}{\partial t}\right|_\mathrm{c} = \frac{nq^2}{4\pi\epsilon_0 m}\int \mathrm{d}^3r' \left(\nabla_\mathbf{r}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\nabla_\mathbf{v}\int \mathrm{d}^3v'\, G(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}',t). \tag{211}$$

- Integrating over velocity[3] (0th moment),

$$\partial_t \int \mathrm{d}^3v\, F + \int \mathrm{d}^3v\, \mathbf{v}\cdot\nabla F + \frac{q}{m}\int \mathrm{d}^3v\, \left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_\mathbf{v} F = \int \mathrm{d}^3v\, \left.\frac{\partial F}{\partial t}\right|_\mathrm{c}$$

yields with the *particle density*[4]

$$n(\mathbf{r},t) = \frac{N}{V}\int \mathrm{d}^3v\, F(\mathbf{r},\mathbf{v},t) \tag{212}$$

that

$$\partial_t n(\mathbf{r},t) + \frac{N}{V}\underbrace{\int \mathrm{d}^3v\, \mathbf{v}\cdot\nabla F}_{\nabla\cdot\int \mathrm{d}^3v\,\mathbf{v}F} = 0.$$

[E] Show that integrals of the type $\int \mathrm{d}^3v\, \nabla_\mathbf{v} g$ or $\int \mathrm{d}^3v\, [\mathbf{v}\times\mathbf{h}(\mathbf{r},t)]\cdot\nabla_\mathbf{v} g$ vanish if $\lim_{|\mathbf{v}|\to\infty} g = 0$.

- Hence we find, not surprisingly, the *continuity equation*

$$\partial_t n(\mathbf{r},t) + \nabla\cdot\underbrace{\mathbf{J}(\mathbf{r},t)}_{n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)} = 0 \tag{213}$$

for particle density and *particle flux*

$$\mathbf{J}(\mathbf{r},t) = \frac{N}{V}\int \mathrm{d}^3v\, \mathbf{v}F(\mathbf{r},\mathbf{v},t) = n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) \tag{214}$$

or *fluid velocity*

$$\mathbf{u}(\mathbf{r},t) = \frac{\mathbf{J}(\mathbf{r},t)}{n(\mathbf{r},t)}. \tag{215}$$

[2] For brevity, the particle species index σ is suppressed here.
[3] We drop the subscript $\mathbf{r}$ in $\nabla_\mathbf{r}$.
[4] Here space and time dependent! Previously n was just the average density N/V.

- In terms of *charge density*

$$\rho(\mathbf{r},t) = qn(\mathbf{r},t) \tag{216}$$

and *current density*

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{J}(\mathbf{r},t) \tag{217}$$

the continuity equation reads

$$\partial_t\rho(\mathbf{r},t) + \nabla\cdot\mathbf{j}(\mathbf{r},t) = 0. \tag{218}$$

E Why is the continuity equation fulfilled as long as there are only collisions on the right hand side of (210)? Which processes would spoil the simple form of the continuity equation?

- Now we multiply (210) by $\frac{N}{V}m\mathbf{v}$ and integrate over $\mathbf{v}$ (1st moment),

$$\frac{N}{V}m\int \mathrm{d}^3v\,\mathbf{v}\left[\partial_t + \mathbf{v}\cdot\nabla + \frac{q}{m}\left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_{\mathbf{v}}\right]F(\mathbf{r},\mathbf{v},t) = \frac{N}{V}m\int \mathrm{d}^3v\,\mathbf{v}\,\frac{\partial F}{\partial t}\bigg|_{\mathrm{c}}.$$

- The first term gives[5]

$$\frac{N}{V}m\int \mathrm{d}^3v\,\mathbf{v}\partial_t F = m\partial_t\mathbf{J} = mn\partial_t\mathbf{u} + m\mathbf{u}\partial_t n = mn\partial_t\mathbf{u} - m\mathbf{u}\nabla\cdot[n\mathbf{u}].$$

In the last step we used the continuity equation.

- The second term reads

$$\begin{aligned}
\frac{N}{V}m\int \mathrm{d}^3v\,\mathbf{v}\mathbf{v}\cdot\nabla F &= \frac{N}{V}m\nabla\cdot\int \mathrm{d}^3v\,\mathbf{v}F\mathbf{v} \\
&= \frac{N}{V}m\nabla\cdot\int \mathrm{d}^3v\,\mathbf{v}F[\mathbf{v}-\mathbf{u}] + \frac{N}{V}m\nabla\cdot\int \mathrm{d}^3v\,[\mathbf{v}F]\mathbf{u} \\
&= \frac{N}{V}m\nabla\cdot\int \mathrm{d}^3v\,\mathbf{v}[\mathbf{v}-\mathbf{u}]F + m\nabla\cdot\mathbf{u}n\mathbf{u} \\
&= \frac{N}{V}m\nabla\cdot\int \mathrm{d}^3v\,[\mathbf{v}-\mathbf{u}][\mathbf{v}-\mathbf{u}]F \\
&\quad + \frac{N}{V}m\nabla\cdot\int \mathrm{d}^3v\,\mathbf{u}[\mathbf{v}-\mathbf{u}]F \\
&\quad + m\nabla\cdot\mathbf{u}n\mathbf{u}.
\end{aligned}$$

The first term on the right hand side is the gradient of the *pressure tensor*

$$\mathsf{P}(\mathbf{r},t) = \frac{N}{V}m\int \mathrm{d}^3v\,[\mathbf{v}-\mathbf{u}(\mathbf{r},t)][\mathbf{v}-\mathbf{u}(\mathbf{r},t)]F(\mathbf{r},\mathbf{v},t), \tag{219}$$

[5] n, $\mathbf{J}$, and $\mathbf{u}$ are functions of $\mathbf{r}$ and t.

i.e., in components

$$\mathsf{P}_{ij}(\mathbf{r},t) = \frac{N}{V} m \int \mathrm{d}^3 v \, [v_i - u_i(\mathbf{r},t)][v_j - u_j(\mathbf{r},t)] F(\mathbf{r},\mathbf{v},t).$$

The other two terms combine to

$$\begin{aligned} & \frac{N}{V} m \nabla \cdot \int \mathrm{d}^3 v \, \mathbf{u}[\mathbf{v}-\mathbf{u}]F + m\nabla\cdot\mathbf{u}n\mathbf{u} \\ = \; & \frac{N}{V} m \nabla \cdot \mathbf{u} \int \mathrm{d}^3 v \, [\mathbf{v}-\mathbf{u}]F + m\nabla\cdot\mathbf{u}n\mathbf{u} \\ = \; & m\nabla\cdot\mathbf{u}n\mathbf{u} - m\nabla\cdot\mathbf{u}n\mathbf{u} + m\nabla\cdot\mathbf{u}n\mathbf{u} = m\nabla\cdot\mathbf{u}n\mathbf{u}. \end{aligned}$$

Hence

$$\frac{N}{V} m \int \mathrm{d}^3 v \, \mathbf{v}\mathbf{v}\cdot\nabla F \quad = \quad \nabla\cdot\mathsf{P} + m\nabla\cdot\mathbf{u}[n\mathbf{u}].$$

E If you are confused you may do the calculation using indices and the sum convention, i.e., $\mathbf{v}\mathbf{v}\cdot\nabla F = v_j v_i \frac{\partial}{\partial x_i} F = \frac{\partial}{\partial x_i} v_j v_i F = \cdots$.

- The third term yields[6]

$$\frac{N}{V} q \int \mathrm{d}^3 v \, \mathbf{v} \left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_{\mathbf{v}} F \quad = \quad -\frac{N}{V} q \int \mathrm{d}^3 v \, F \nabla_{\mathbf{v}} \cdot \left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\mathbf{v}.$$

The $\overline{\mathbf{E}}$-part gives (sum rule implied)

$$F\frac{\partial}{\partial v_i}(\overline{E}_i v_j) = F\overline{E}_i\delta_{ij} \; = F\overline{E}_j = F\overline{\mathbf{E}}\cdot\mathbf{e}_j \;,$$

the $\mathbf{v}\times\overline{\mathbf{B}}$-part

$$F\frac{\partial}{\partial v_i}\epsilon_{ikl}v_k\overline{B}_l v_j = F\epsilon_{ikl}\overline{B}_l(\delta_{ik}v_j + v_k\delta_{ij}) = F\epsilon_{ikl}\overline{B}_l v_k\delta_{ij} = F\epsilon_{jkl}\overline{B}_l v_k = F(\mathbf{v}\times\overline{\mathbf{B}})\cdot\mathbf{e}_j$$

so that

$$\frac{N}{V} q \int \mathrm{d}^3 v \, \mathbf{v} \left(\overline{\mathbf{E}} + \mathbf{v}\times\overline{\mathbf{B}}\right)\cdot\nabla_{\mathbf{v}} F = -qn(\overline{\mathbf{E}} + \mathbf{u}\times\overline{\mathbf{B}}).$$

- Putting everything together yields

$$mn\partial_t\mathbf{u} \underbrace{-m\mathbf{u}\nabla\cdot[n\mathbf{u}] + m\nabla\cdot\mathbf{u}[n\mathbf{u}]}_{nm\mathbf{u}\cdot\nabla\mathbf{u}} + \nabla\cdot\mathsf{P} - qn(\overline{\mathbf{E}} + \mathbf{u}\times\overline{\mathbf{B}}) = \frac{N}{V} m \int \mathrm{d}^3 v \, \mathbf{v} \left.\frac{\partial F}{\partial t}\right|_{\mathrm{c}}.$$

[6] The surface integral in velocity space vanishes.

- If there were just one fluid species the right hand side has to vanish (without fluid sinks or sources), and we obtain NEWTONS equation of motion for a fluid (in the absence of dissipative effects)

$$\begin{aligned} mn(\mathbf{r},t)\,[\partial_t + \mathbf{u}(\mathbf{r},t)\cdot\nabla]\,\mathbf{u}(\mathbf{r},t) \\ = qn(\mathbf{r},t)[\overline{\mathbf{E}}(\mathbf{r},t) + \mathbf{u}(\mathbf{r},t)\times\overline{\mathbf{B}}(\mathbf{r},t)] - \nabla\cdot\mathsf{P}(\mathbf{r},t), \end{aligned} \tag{220}$$

  also called EULER equation. The right hand side in this equation is a *force density* with the *pressure gradient force* contributing. Note the origin of this pressure term: it arose from the $\mathbf{v}\cdot\nabla F$-term due to a difference between fluid velocity $\mathbf{u}$ and particle velocity $\mathbf{v}$.

- The derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \mathbf{u}(\mathbf{r},t)\cdot\nabla \tag{221}$$

  is called the *material derivative*, the *comoving derivative*, the *convective derivative* or the *Lagrangian derivative*.

- The pressure tensor is commonly split into a diagonal part and an off-diagonal part,

$$\mathsf{P}(\mathbf{r},t) = \mathsf{I}\,p(\mathbf{r},t) + \Pi(\mathbf{r},t) \tag{222}$$

  (I is the 3×3 unity matrix) with the scalar *pressure*

$$p(\mathbf{r},t) = \frac{N}{V}\frac{m}{3}\int \mathrm{d}^3v\,[\mathbf{v}-\mathbf{u}(\mathbf{r},t)]^2 F(\mathbf{r},\mathbf{v},t) = n(\mathbf{r},t)k_{\mathrm{B}}T(\mathbf{r},t), \tag{223}$$

  where $T(\mathbf{r},t)$ is the local *fluid temperature*, and

$$\Pi(\mathbf{r},t) = \mathsf{P}(\mathbf{r},t) - \mathsf{I}\,p(\mathbf{r},t) \tag{224}$$

  is the *viscosity tensor*.

- Note that by the *definition* of a temperature $T(\mathbf{r},t)$ according (223) we establish locally something that looks like the equation of state of an ideal gas.

[E] If collisions between the same particle species occur sufficiently frequently the distribution function is isotropic in velocity space. Show that then $\mathsf{P}(\mathbf{r},t) = \mathsf{I}\,p(\mathbf{r},t)$ and $\nabla\cdot\mathsf{P} = \nabla p$.

- In the case of various particle (i.e., fluid) species we have instead of (220) the Euler equation

$$m_\sigma n_\sigma[\partial_t + \mathbf{u}_\sigma \cdot \nabla]\mathbf{u}_\sigma - q_\sigma n_\sigma(\overline{\mathbf{E}} + \mathbf{u}_\sigma \times \overline{\mathbf{B}}) + \nabla \cdot \mathsf{P}_\sigma \simeq -m_\sigma n_\sigma \sum_{\sigma'} \langle \nu_{\sigma\sigma'} \rangle (\mathbf{u}_\sigma - \mathbf{u}_{\sigma'}), \qquad (225)$$

where $\langle \nu_{\sigma\sigma'} \rangle$ is an *effective collision frequency* that allows for momentum transfer from fluid σ' to fluid σ due to collisions between the respective particle species.[7]

E Derive the *fluid energy transport equation* by multiplying (210) with $\frac{N}{2V} m \mathbf{v}^2$ and integrating over $\mathbf{v}$ (2nd moment).

- Although we started with the electrostatic approximation and just one particle species it is clear by now that in a fluid description the continuity and momentum balance equations (for each species σ) are to be solved together with Maxwell equations using $\rho(\mathbf{r}, t) = \sum_\sigma \rho_\sigma(\mathbf{r}, t) + \rho_0(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t) = \sum_\sigma \mathbf{j}_\sigma(\mathbf{r}, t) + \mathbf{j}_0(\mathbf{r}, t)$ as the sources for the fields $\overline{\mathbf{E}}(\mathbf{r}, t)$ and $\overline{\mathbf{B}}(\mathbf{r}, t)$. Here, as in (143) and (145), ρ_0 and $\mathbf{j}_0$ are charge density and current density formally generating the given external fields.

4.2 TWO-FLUID EQUATIONS

- We consider an electron and an ion fluid ($\sigma =$e,i) governed by continuity

$$\partial_t n_\sigma + \nabla \cdot (n_\sigma \mathbf{u}_\sigma) = 0, \qquad (226)$$

momentum balance

$$m_\sigma n_\sigma[\partial_t + \mathbf{u}_\sigma \cdot \nabla]\mathbf{u}_\sigma - q_\sigma n_\sigma(\overline{\mathbf{E}} + \mathbf{u}_\sigma \times \overline{\mathbf{B}}) + \nabla p_\sigma + \nabla \cdot \Pi_\sigma \simeq -m_\sigma n_\sigma \sum_{\sigma'} \langle \nu_{\sigma\sigma'} \rangle (\mathbf{u}_\sigma - \mathbf{u}_{\sigma'}), \qquad (227)$$

and Maxwell equations

$$\epsilon_0 \nabla \cdot \overline{\mathbf{E}} = \rho_0 + \sum_\sigma \rho_\sigma, \qquad \rho_\sigma = q_\sigma n_\sigma, \qquad (228)$$

$$\nabla \times \overline{\mathbf{B}} = \mu_0 \left(\mathbf{j}_0 + \sum_\sigma \mathbf{j}_\sigma \right) + \mu_0 \epsilon_0 \partial_t \overline{\mathbf{E}}, \qquad \mathbf{j}_\sigma = q_\sigma n_\sigma \mathbf{u}_\sigma \qquad (229)$$

[7] The actual calculation of $\langle \nu_{\sigma\sigma'} \rangle$ from kinetic equations is rather involved. If time permits, we will discuss this in chapter "Kinetic Description of Plasma II".

$$\nabla \cdot \overline{\mathbf{B}} = 0, \tag{230}$$
$$\nabla \times \overline{\mathbf{E}} = -\partial_t \overline{\mathbf{B}}. \tag{231}$$

- The coupling between electrons and ions occurs collisionless via MAXWELL equations and momentum transfer because of collisions (i.e., $\langle \nu_{\sigma\sigma'} \rangle$).

- In order to solve the set of equations one needs to assume a relation between pressure and density. Otherwise we encounter the same kind of hierarchy problem as in BBGKY: the continuity equation couples n to $\mathbf{u}$, the momentum equation couples $\mathbf{u}$ to P [or at least p (or T)]. The energy transport equation couples kinetic energy density nmu^2 and thermal energy density $3nk_BT/2$ to higher-moment entities, etc.

- Which relation between pressure and density is reasonable to choose depends on the physical process to be described. If heat flow is so fast that temperature equilibrates quickly one has the isothermal relationship

  $$p \sim n \qquad \text{(isothermal)}.$$

  If heat flow is slow compared to the process studied one has

  $$p \sim n^\gamma, \quad \gamma = \frac{2+f}{f} \qquad \text{(adiabatic)}$$

  where f is the number of degrees of freedom.

Electron plasma wave from the fluid perspective

- As an example we consider a one-dimensional electrostatic perturbation $n^{(1)}$ in the electron fluid $n = n_e = n^{(0)} + n^{(1)}$ of an otherwise homogeneous plasma.

- Assuming an adiabatic relationship between pressure and density (where in 1D $\gamma = 3$) we have

  $$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n} \tag{232}$$

  so that with (223) $p = nk_BT$ in 1D

  $$\nabla p = 3k_BT\nabla n = 3k_BT\nabla(n^{(0)} + n^{(1)}) = 3k_BT\partial_x n^{(1)}\,\mathbf{e}_x.$$

- The linearized momentum balance (227) (neglecting collisions and viscosity) reduces to

$$mn^{(0)}\partial_t \mathbf{u}^{(1)} + en^{(0)}\overline{\mathbf{E}}^{(1)} + 3k_\mathrm{B}T\partial_x n^{(1)}\,\mathbf{e}_x = 0.$$

- Making the ansatz

$$\mathbf{u}^{(1)} = \hat{u}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,\mathbf{e}_x, \qquad \overline{\mathbf{E}}^{(1)} = \hat{E}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,\mathbf{e}_x, \qquad n^{(1)} = \hat{n}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,\mathbf{e}_x$$

  we obtain

$$-\mathrm{i}\omega mn^{(0)}\hat{u} + en^{(0)}\hat{E} + 3\mathrm{i}k_\mathrm{B}Tk\hat{n} = 0. \tag{233}$$

- The linearized continuity equation

$$\partial_t n^{(1)} + \nabla\cdot\left(n^{(0)}\mathbf{u}^{(1)}\right) = 0$$

  gives

$$-\mathrm{i}\omega\hat{n} + \mathrm{i}n^{(0)}k\hat{u} = 0 \quad \Rightarrow \quad \hat{n} = \frac{n^{(0)}k}{\omega}\,\hat{u}$$

  while the Maxwell equation (228) gives

$$\mathrm{i}\epsilon_0 k\hat{E} = -e\hat{n} \quad \Rightarrow \quad \hat{E} = \frac{\mathrm{i}en^{(0)}}{\epsilon_0\omega}\,\hat{u}.$$

- Plugging this into (233) yields

$$\left(-\mathrm{i}\omega mn^{(0)} + en^{(0)}\frac{\mathrm{i}en^{(0)}}{\epsilon_0\omega} + 3\mathrm{i}k_\mathrm{B}Tk\frac{n^{(0)}k}{\omega}\right)\hat{u} = 0$$

  so that

$$\omega^2 = \frac{e^2n^{(0)}}{\epsilon_0 m} + 3\frac{k_\mathrm{B}T}{m}k^2 = \omega_\mathrm{p}^2 + \frac{3}{2}k^2v_\mathrm{th}^2 = \omega_\mathrm{p}^2\left(1 + 3k^2\lambda_\mathrm{D}^2\right).$$

- We thus recover the Bohm-Gross dispersion relation (200) with very little effort. However, in the "fluid way" pursued here we do not get the imaginary part of ω (i.e., Landau damping), and we had to *assume* the adiabatic relation (232). Instead, in the more involved kinetic approach we did not need to assume any equation of state.

Ion acoustic wave revisited

- For the 1D, linearized, ionic momentum balance equation we have (with $m_\mathrm{i} = M$, $\gamma_\mathrm{i} = 3$)

$$Mn_\mathrm{i}^{(0)}\partial_t u_\mathrm{i}^{(1)} - en_\mathrm{i}^{(0)} \underbrace{E^{(1)}}_{-\partial_x\Phi^{(1)}} + 3k_\mathrm{B}T_\mathrm{i}\partial_x n_\mathrm{i}^{(1)} = 0.$$

A plane-wave ansatz as above yields

$$-\mathrm{i}\omega M n_\mathrm{i}^{(0)} u_\mathrm{i}^{(1)} = -en_\mathrm{i}^{(0)}\mathrm{i}k\Phi^{(1)} - 3k_\mathrm{B}T_\mathrm{i}\mathrm{i}kn_\mathrm{i}^{(1)}. \tag{234}$$

- Because the electrons are fast on the ionic time scale
  - we use the isothermal $p(n)$-dependence, i.e., $\gamma_\mathrm{e} = 1$,
  - $n_\mathrm{e} = n_\mathrm{i} = n_\mathrm{i}^{(0)} + n_\mathrm{i}^{(1)} = n$, and
  - we set $m = m_\mathrm{e} = 0$ in the momentum balance equation (i.e., neglect of electron inertia),

$$en \underbrace{E^{(1)}}_{-\partial_x\Phi^{(1)}} + k_\mathrm{B}T_\mathrm{e}\partial_x n = 0 \quad \Rightarrow \quad e\partial_x\Phi^{(1)} = k_\mathrm{B}T_\mathrm{e}\,\frac{\partial_x n}{n} \quad \Rightarrow \quad n = n_0\; \mathrm{e}^{e\Phi^{(1)}/(k_\mathrm{B}T_\mathrm{e})}$$

(Debye screening).

- Expanding for small $e\Phi^{(1)}/(k_\mathrm{B}T_\mathrm{e})$

$$n = n_0 + \frac{en_0\Phi^{(1)}}{k_\mathrm{B}T_\mathrm{e}}$$

we can directly read off

$$n_\mathrm{i}^{(0)} = n_0, \qquad n_\mathrm{i}^{(1)} = \frac{en_0\Phi^{(1)}}{k_\mathrm{B}T_\mathrm{e}}. \tag{235}$$

- This allows to get rid of $\Phi^{(1)}$ in (234),

$$\omega M n_0 u_\mathrm{i}^{(1)} = k\left(k_\mathrm{B}T_\mathrm{e} + 3k_\mathrm{B}T_\mathrm{i}\right) n_\mathrm{i}^{(1)}.$$

- $u_\mathrm{i}^{(1)}$ can be expressed in terms of $n_\mathrm{i}^{(1)}$ using continuity,

$$-\mathrm{i}\omega n_\mathrm{i}^{(1)} + \mathrm{i}kn_0 u_\mathrm{i}^{(1)} = 0 \quad \Rightarrow \quad u_\mathrm{i}^{(1)} = \frac{\omega}{kn_0}\, n_\mathrm{i}^{(1)},$$

so that

$$\omega^2 = \frac{k^2}{M}\left(k_\mathrm{B}T_\mathrm{e} + 3k_\mathrm{B}T_\mathrm{i}\right).$$

- For $T_\mathrm{e}/T_\mathrm{i} \gg 1$ follows
$$\omega^2 \simeq k^2 C_s^2,$$
i.e., we recover (206), (208) in the long-wavelength limit $k^2\lambda_\mathrm{D}^2 \ll 1$.

4.3 ONE-FLUID EQUATIONS

- Let us add the continuity equation (226) for electrons and ions (each multiplied by the respective mass),
$$\partial_t(mn_\mathrm{e} + Mn_\mathrm{i}) + \nabla\cdot(mn_\mathrm{e}\mathbf{u}_\mathrm{e} + Mn_\mathrm{i}\mathbf{u}_\mathrm{i}) = 0.$$
- Introducing the *total mass density*
$$\rho_m(\mathbf{r},t) = \sum_\sigma m_\sigma n_\sigma(\mathbf{r},t) = mn_\mathrm{e}(\mathbf{r},t) + Mn_\mathrm{i}(\mathbf{r},t) \tag{236}$$
and the *center-of-mass velocity*
$$\mathbf{U}(\mathbf{r},t) = \frac{\sum_\sigma m_\sigma n_\sigma(\mathbf{r},t)\mathbf{u}_\sigma(\mathbf{r},t)}{\rho_m(\mathbf{r},t)} = \frac{mn_\mathrm{e}(\mathbf{r},t)\mathbf{u}_\mathrm{e}(\mathbf{r},t) + Mn_\mathrm{i}(\mathbf{r},t)\mathbf{u}_\mathrm{i}(\mathbf{r},t)}{\rho_m(\mathbf{r},t)} \tag{237}$$
this becomes
$$\boxed{\partial_t\rho_m(\mathbf{r},t) + \nabla\cdot[\rho_m(\mathbf{r},t)\mathbf{U}(\mathbf{r},t)] = 0}\,. \tag{238}$$
- If we add the continuity equation (226) for electrons and ions after multiplying them with their respective charge $q_\mathrm{e} = -e$, $q_\mathrm{i} = e$,
$$\partial_t(-en_\mathrm{e} + en_\mathrm{i}) + \nabla\cdot(-en_\mathrm{e}\mathbf{u}_\mathrm{e} + en_\mathrm{i}\mathbf{u}_\mathrm{i}) = 0,$$
we obtain, with the *total charge density*
$$\rho(\mathbf{r},t) = \sum_\sigma q_\sigma n_\sigma(\mathbf{r},t) = e[n_\mathrm{i}(\mathbf{r},t) - n_\mathrm{e}(\mathbf{r},t)] \tag{239}$$
and the *total current density*
$$\mathbf{j}(\mathbf{r},t) = \sum_\sigma q_\sigma n_\sigma(\mathbf{r},t)\mathbf{u}_\sigma(\mathbf{r},t) = -en_\mathrm{e}(\mathbf{r},t)\mathbf{u}_\mathrm{e}(\mathbf{r},t) + en_\mathrm{i}(\mathbf{r},t)\mathbf{u}_\mathrm{i}(\mathbf{r},t), \tag{240}$$
the “usual”
$$\boxed{\partial_t\rho(\mathbf{r},t) + \nabla\cdot\mathbf{j}(\mathbf{r},t) = 0}\,. \tag{241}$$

- Deriving the one-fluid EULER equation is a bit more cumbersome. The collision terms for collisions between electrons and ions have to cancel each other in the center-of-mass frame.[8] Expressing everything in terms of ρ_m, ρ, $\mathbf{U}$, $\mathbf{j}$, and

$$\mathsf{P}(\mathbf{r},t) = \sum_\sigma \mathsf{P}_\sigma^{\mathrm{CM}}(\mathbf{r},t) = \mathsf{P}_\mathrm{e}^{\mathrm{CM}}(\mathbf{r},t) + \mathsf{P}_\mathrm{i}^{\mathrm{CM}}(\mathbf{r},t), \tag{242}$$

$$\mathsf{P}_\sigma^{\mathrm{CM}}(\mathbf{r},t) = \frac{N_\sigma}{V} m_\sigma \int \mathrm{d}^3v\, [\mathbf{v} - \mathbf{U}(\mathbf{r},t)][\mathbf{v} - \mathbf{U}(\mathbf{r},t)] F_\sigma(\mathbf{r},\mathbf{v},t) \tag{243}$$

($\mathbf{U}$ inside instead of $\mathbf{u}_\sigma$) we find the EULER equation [suppressing all arguments $(\mathbf{r},t)$]

$$\boxed{\rho_m \partial_t \mathbf{U} + \rho_m \mathbf{U}\cdot\nabla\mathbf{U} = \rho\overline{\mathbf{E}} + \mathbf{j}\times\overline{\mathbf{B}} - \nabla\cdot\mathsf{P}}\,. \tag{244}$$

E Derive (244). It may be quicker to restart from taking the 1st moment of (210) (but smuggling-in $\mathbf{U}$ instead of $\mathbf{u}$) instead of rewriting the sum of the two-fluid EULER equations in terms of ρ_m, ρ, $\mathbf{U}$, $\mathbf{j}$, and P.

4.3.1 OHM's *law*

- Multiplying the EULER equations for electrons and ions (227) by q_σ/m_σ, adding them, and expressing as much as possible in terms of ρ_m, ρ, $\mathbf{U}$, $\mathbf{j}$, and $\mathsf{P}_\sigma^{\mathrm{CM}}$ yields *generalized* OHM's *law*

$$\boxed{\begin{aligned} \partial_t\mathbf{j} + \nabla\cdot(\mathbf{U}\mathbf{j} + \mathbf{j}\mathbf{U} - \mathbf{U}\mathbf{U}\rho) \;=\;& e^2\left(\frac{n_\mathrm{e}}{m} + \frac{n_\mathrm{i}}{M}\right)\overline{\mathbf{E}} \\ &+\frac{e^2}{m+M}\left(\frac{1}{m} + \frac{1}{M}\right)\rho_m\mathbf{U}\times\overline{\mathbf{B}} \\ &-\frac{e}{m+M}\left(\frac{M}{m} - \frac{m}{M}\right)\mathbf{j}\times\overline{\mathbf{B}} \\ &-\frac{e}{m}\nabla\cdot\left(\frac{m}{M}\mathsf{P}_\mathrm{i}^{\mathrm{CM}} - \mathsf{P}_\mathrm{e}^{\mathrm{CM}}\right) - \nu\mathbf{j} \end{aligned}} \tag{245}$$

where

$$-\nu\mathbf{j} \simeq \sum_\sigma \int \mathrm{d}^3v\, \frac{N_\sigma}{V}\mathbf{v}\, \frac{\partial F_\sigma}{\partial t}\bigg|_\mathrm{c}, \tag{246}$$

with ν an average collision frequency.

[8] If there are collisions with a third species, e.g., neutral atoms, a dissipative term survives, however.

- The relations between ν and the electrical *resistivity* η and *conductivity* σ are

$$\eta = \frac{\nu m}{n_e e^2}, \qquad \sigma = \frac{1}{\eta}. \tag{247}$$

- As required, in a static, uniform, neutral, unmagnetized system only

$$\nu \mathbf{j} = e^2 \left(\frac{n_e}{m} + \frac{n_i}{M} \right) \overline{\mathbf{E}} \simeq \frac{e^2 n_e}{m} \overline{\mathbf{E}}$$

$$\Rightarrow \qquad \mathbf{j} = \sigma \overline{\mathbf{E}}, \tag{248}$$

remains.

4.3.2 *Magnetohydrodynamics (MHD)*

- The set of one-fluid equations (238), (241), (244), and (245) is still quite involved (and, in fact, contains as much information as the set of two-fluid equations). Further simplifications are possible under certain circumstances:[9]
  - If phenomena on length scales $L \gg \lambda_D$ are studied, quasineutrality and the continuity equation require that

$$\frac{\rho_e - \rho_i}{\rho_e} \ll 1 \qquad \Rightarrow \qquad \nabla \cdot \mathbf{j} = 0.$$

  - From our study of drift motions in chapter 2 we expect that in a magnetized plasma the assumption $n_e = n_i$ is only valid on length scales larger than the biggest Larmor radius and on time scales large compared to the biggest inverse cyclotron frequency,

$$\frac{r_{L,i}}{L} \ll 1, \qquad T\omega_{c,i} \gg 1.$$

  - If phenomena on large length scales L (and slow time-scales T) are considered we have because of the Maxwell equation $\nabla \times \overline{\mathbf{B}} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \partial_t \overline{\mathbf{E}} \simeq \mu_0 \mathbf{j}$ that $\mathbf{j} \times \overline{\mathbf{B}} \sim \frac{B^2}{L}$, which may allow for neglecting

[9] Going through all the possible cases without a concrete problem at hand is a little bit too much of dry swimming for a lecture. We therefore restrict ourselves to a few important aspects.

the $\mathbf{j} \times \overline{\mathbf{B}}$ compared to the $\rho_m \mathbf{U} \times \overline{\mathbf{B}}$-term in OHM's law (245), leading [with the otherwise same assumptions that led to (248)]

$$\mathbf{j} = \sigma \left(\overline{\mathbf{E}} + \mathbf{U} \times \overline{\mathbf{B}} \right). \tag{249}$$

This can be easily understood, as $\overline{\mathbf{E}}' = \overline{\mathbf{E}} + \mathbf{U} \times \overline{\mathbf{B}}$ is the LORENTZ-transformed electric field "seen" by the moving fluid element in its rest frame.

– If $\sigma \to \infty$ OHM's law reduces to

$$\overline{\mathbf{E}} + \mathbf{U} \times \overline{\mathbf{B}} = \mathbf{0}. \tag{250}$$

– In perturbative calculations often $\mathbf{U}^{(0)} = \mathbf{0}$ in a properly chosen system of reference so that terms $\sim \mathbf{U} \cdot \nabla \mathbf{U}$ or $\sim \nabla \cdot \mathbf{j} \mathbf{U}$ etc. are of second order and thus can be neglected.

– Terms $\sim m/M$ can often be neglected. This then implies $\rho_m \simeq M n_\mathrm{i}$.

– If collisions between particles of species σ are frequent enough on the time-scale of interest, the assumption of an isotropic pressure may be reasonable,[10] i.e., $\nabla \cdot \mathsf{P}_\sigma \simeq \nabla p_\sigma$.

- Let us summarize now the simplified set of one-fluid equations, the so-called *MHD equations*

$$\boxed{\begin{aligned} \partial_t \rho_m + \nabla \cdot [\rho_m \mathbf{U}] &= 0, \\ \rho_m \partial_t \mathbf{U} &= \mathbf{j} \times \overline{\mathbf{B}} - \nabla p, \\ \overline{\mathbf{E}} + \mathbf{U} \times \overline{\mathbf{B}} &= \eta \mathbf{j} + \left(\frac{\nabla p_\mathrm{i}}{e n_\mathrm{e}} \right), \\ \nabla \times \overline{\mathbf{E}} &= -\partial_t \overline{\mathbf{B}}, \\ \nabla \times \overline{\mathbf{B}} &= \mu_0 \mathbf{j}. \end{aligned}} \tag{251}$$

- The term in the bracket, $\frac{\nabla p_\mathrm{i}}{e n_\mathrm{e}}$, is often not included in the set of MHD equations but we keep it for the examples below.

4.3.3 *Properties of MHD plasma*

- In a stationary situation the momentum balance eq. in (251) reduces to

$$\mathbf{j} \times \overline{\mathbf{B}} = \nabla p, \tag{252}$$

which implies that ∇p is perpendicular to both $\mathbf{j}$ and $\overline{\mathbf{B}}$.

[10] In NAVIER-STOKES theory this is quantified by the REYNOLDS number.

- In other words, magnetic field lines and charge density currents do not pass *isobaric surfaces*. One may as well state that the magnetic field lines *define* a magnetic surface, which equals an isobaric surface, and charge currents do not pass this surface in a stationary setup.

- The field lines on such a surface do not need to have a constant density, i.e., $\overline{B}$ is, in general, not constant. Instead, we have

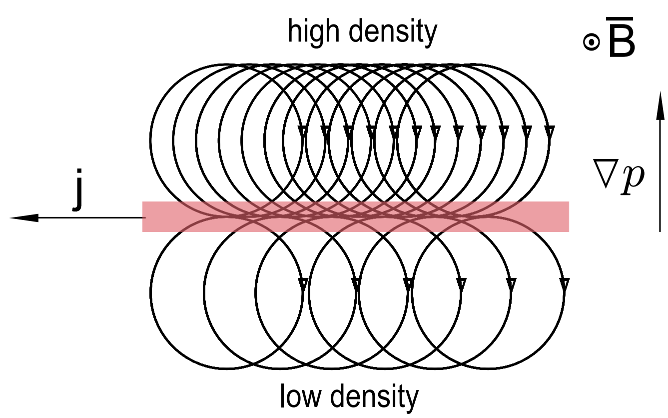
$$\mu_0 \mathbf{j} \times \overline{\mathbf{B}} = (\nabla \times \overline{\mathbf{B}}) \times \overline{\mathbf{B}} = -\frac{1}{2}\nabla \overline{B}^2 + \overline{\mathbf{B}} \cdot \nabla \overline{\mathbf{B}} = \mu_0 \nabla p. \qquad (253)$$

  Hence, on an isobaric surface $\frac{1}{2}\nabla \overline{B}^2 = \overline{\mathbf{B}} \cdot \nabla \overline{\mathbf{B}}$.

- A current $\mathbf{j}_\perp$ perpendicular to $\overline{\mathbf{B}}$ only flows if the pressure gradient does not vanish because from (252) follows

$$\mathbf{j}_\perp = \frac{\overline{\mathbf{B}} \times \nabla p}{B^2}.$$

- Consider the following setup, which explains the microscopic origin of a net current without mass transport:



  A net current $\mathbf{j}$ remains after averaging over the cyclotron motion because there are more plasma particles in the higher-density region than in the lower-density region. This current is perpendicular to both $\overline{\mathbf{B}}$ and ∇p, i.e., it flows on an isobaric surface perpendicular to the magnetic field lines.

  Note that there is a net $\mathbf{j} \neq \mathbf{0}$ despite a net $\mathbf{U} = \mathbf{0}$, i.e., we have no mass flow but nevertheless a current.

E Consider a stationary, current-free, magnetized plasma with constant pressure. Show that, with an electric field $\overline{\mathbf{E}}$ present, a net mass flow

$$\mathbf{U}_\perp = \frac{\overline{\mathbf{E}} \times \overline{\mathbf{B}}}{B^2} \tag{254}$$

across a magnetic field is possible.

- This is the opposite situation: mass flow $\mathbf{U}$ without current $\mathbf{j}$.
- This is also a method to *generate* an electric field $\overline{\mathbf{E}} = -\mathbf{U} \times \overline{\mathbf{B}}$ from a plasma that flows through magnetic field lines, i.e., kinetic energy of a plasma is converted to electric field energy.

- From (253) follows

$$\nabla \left(p + \frac{\overline{B}^2}{2\mu_0} \right) = \frac{\overline{\mathbf{B}} \cdot \nabla \overline{\mathbf{B}}}{\mu_0}. \tag{255}$$

If the right hand side vanishes[11] we have $\nabla p = -\frac{1}{2\mu_0}\nabla \overline{B}^2$, which shows that *magnetic fields exert pressure on plasma*.

4.3.4 *Equilibrium pinch*

- We are looking for a solution of the stationary MHD equations where the magnetic pressure balances the particle pressure.
- Consider the cylindrically symmetric setup of a plasma column for $r < R$, no external fields, and a uniform current density in $\mathbf{e}_z$-direction,

$$\mathbf{j} = j_z \mathbf{e}_z.$$

- We expect an azimuthal magnetic field $B_\varphi(r)\mathbf{e}_\varphi$. It is easy to check that the LORENTZ force on the plasma particles acts inwards (for both $q > 0$ and $q < 0$). Hence we expect a contraction of the plasma column until the magnetic pressure $\sim B^2$ is balanced by the thermodynamic pressure p. In the following we are considering this time instant of idealized, perfect balance so that we can use the stationary set of MHD equations.

[11] As it does, e.g., for $\overline{\mathbf{B}} = \overline{B}(x, y)\mathbf{e}_z$.

- In steady state eqs. (251) become

$$\frac{1}{r}\partial_r(r\rho_m U_r) = 0, \quad (256)$$

$$\partial_r p = -j_z B_\varphi, \quad (257)$$

$$E_r - U_z B_\varphi = \frac{1}{en}\partial_r p_i, \qquad E_\varphi = 0, \quad E_z = \eta j_z, \quad (258)$$

$$\frac{1}{r}\partial_r(rB_\varphi) = \mu_0 j_z. \quad (259)$$

- The last equation gives, upon integration,

$$B_\varphi(r) = \frac{1}{2}\mu_0 r j_z \qquad r \le R, \qquad\qquad B_\varphi(r) = \frac{1}{2}\mu_0 \frac{R^2}{r} j_z \qquad r > R. \quad (260)$$

- The second equation gives the pressure $p(r) = -\frac{1}{4}\mu_0 r^2 j_z^2 + \text{const}$. Because $p = 0$ for $r > R$ we have

$$p(r) = \frac{1}{4}\mu_0(R^2 - r^2)j_z^2 \qquad r \le R, \qquad\qquad p = 0 \qquad r > R.$$

- From (257) and (259) follows

$$\partial_r p = -B_\varphi \frac{1}{\mu_0 r}\partial_r(rB_\varphi) = -B_\varphi^2 \frac{1}{\mu_0 r} - B_\varphi \frac{1}{\mu_0}\partial_r B_\varphi = -B_\varphi^2 \frac{1}{\mu_0 r} - \frac{1}{2\mu_0}\partial_r B_\varphi^2.$$

  Employing (260), for $r \le R$, we may write

$$B_\varphi^2 \frac{1}{\mu_0 r} = B_\varphi \frac{1}{2} j_z,$$

  and, using (259) again

$$B_\varphi^2 \frac{1}{\mu_0 r} = B_\varphi \frac{1}{2} j_z = B_\varphi \frac{1}{2\mu_0 r}\partial_r(rB_\varphi) = B_\varphi^2 \frac{1}{2\mu_0 r} + \frac{1}{4\mu_0}\partial_r B_\varphi^2 \quad \Rightarrow \quad B_\varphi^2 \frac{1}{2\mu_0 r} = \frac{1}{4\mu_0}\partial_r B_\varphi^2.$$

  so that

$$\partial_r p = -\frac{1}{\mu_0}\partial_r B_\varphi^2$$

  and thus

$$p + \frac{1}{\mu_0}B_\varphi^2 = \text{const.}$$

E Show that this result is consistent with (255).

- The pressure on axis

$$p(0) = \frac{1}{4}\mu_0 R^2 j_z^2$$

defines the equilibrium radius of the pinch

$$R = \frac{1}{j_z}\sqrt{\frac{4p(0)}{\mu_0}}.$$

- The current is (assuming $p(0) = n_e k_B T_e + n_i k_B T_i = 2nk_B T$)

$$I = j_z \pi R^2 = \sqrt{\frac{8nk_B T}{\mu_0}}\,\pi R = \sqrt{\frac{8\pi N_\ell k_B T}{\mu_0}},$$

where $N_\ell = n\pi R^2$ is the number of electrons (or ions) per meter. In order to get an idea about the orders of magnitude involved one may write this as

$$I[\mathrm{A}] = 5.7 \times 10^{-5}\,(N_\ell[\mathrm{m}^{-1}])^{1/2}\,(k_B T[\mathrm{keV}])^{1/2}.$$

- We have not made use of (256) and (258). The assumption in the above derivation was that j_z =const. This may not be a good assumption. In fact, different solutions are found for other assumptions:

E Show that under the assumptions $j_z = -enu_{e,z}$, $u_{e,z}$ = const, T_e = const, T_i = const, the density profile of the pinch is

$$n(r) = \frac{n(r=0)}{(1+br^2)^2},$$

and determine the constant b.

4.3.5 *Magnetic and electric field dynamics*

- The momentum equation (251.2), together with (251.5), gives

$$\boxed{\rho_m \partial_t \mathbf{U} = \frac{1}{\mu_0}(\nabla \times \overline{\mathbf{B}}) \times \overline{\mathbf{B}} - \nabla p}\,. \tag{261}$$

As $(\nabla \times \overline{\mathbf{B}}) \times \overline{\mathbf{B}} = (\overline{\mathbf{B}} \cdot \nabla)\overline{\mathbf{B}} - \frac{1}{2}\nabla \overline{B}^2$, the first term on the right hand side acts on the plasma in such a way to make the magnetic field lines as straight as possible [$(\overline{\mathbf{B}}\cdot\nabla)\overline{\mathbf{B}}$-part] and exerts magnetic pressure ($-\frac{1}{2}\nabla\overline{B}^2$-part).

The curl of Ohm's law (251.3),

$$\nabla\times(\overline{\mathbf{E}}+\mathbf{U}\times\overline{\mathbf{B}}) = \nabla\times\left[\eta\mathbf{j}+\left(\frac{\nabla p_{\mathrm{i}}}{en_{\mathrm{e}}}\right)\right],$$

leads to [using (251.4) and (251.5)]

$$-\partial_t\overline{\mathbf{B}}+\nabla\times(\mathbf{U}\times\overline{\mathbf{B}}) = \frac{\eta}{\mu_0}\nabla\times(\nabla\times\overline{\mathbf{B}})$$

$$\Rightarrow \qquad -\partial_t\overline{\mathbf{B}}+\mathbf{U}(\underbrace{\nabla\cdot\overline{\mathbf{B}}}_{0})-\overline{\mathbf{B}}(\nabla\cdot\mathbf{U})+(\overline{\mathbf{B}}\cdot\nabla)\mathbf{U}-(\mathbf{U}\cdot\nabla)\overline{\mathbf{B}} = \frac{1}{\sigma\mu_0}[\nabla(\underbrace{\nabla\cdot\overline{\mathbf{B}}}_{0})-\nabla^2\overline{\mathbf{B}}]$$

$$\Rightarrow \qquad \boxed{\frac{1}{\sigma\mu_0}\nabla^2\overline{\mathbf{B}}-\partial_t\overline{\mathbf{B}} = \overline{\mathbf{B}}(\nabla\cdot\mathbf{U})-(\overline{\mathbf{B}}\cdot\nabla)\mathbf{U}+(\mathbf{U}\cdot\nabla)\overline{\mathbf{B}}}\,. \qquad (262)$$

This is the equation of motion for the magnetic field.

- The equation of motion for the electric field is obtained by taking the time-derivative of Ohm's law (251.3) (neglecting the pressure term) and using Faraday's law (251.4) and Ampère's law (251.5),

$$\partial_t\overline{\mathbf{E}}+\partial_t(\mathbf{U}\times\overline{\mathbf{B}}) = \eta\partial_t\mathbf{j}$$

$$\Rightarrow \qquad \partial_t\overline{\mathbf{E}}+\partial_t(\mathbf{U}\times\overline{\mathbf{B}}) = -\frac{1}{\sigma\mu_0}\nabla\times(\nabla\times\overline{\mathbf{E}}) = -\frac{1}{\sigma\mu_0}[\nabla(\underbrace{\nabla\cdot\overline{\mathbf{E}}}_{\sim\rho=0})-\nabla^2\overline{\mathbf{E}}]$$

$$\Rightarrow \qquad \boxed{\partial_t\overline{\mathbf{E}}-\frac{1}{\sigma\mu_0}\nabla^2\overline{\mathbf{E}} = -\partial_t(\mathbf{U}\times\overline{\mathbf{B}})}\,. \qquad (263)$$

- Writing this as

$$\partial_t\overline{\mathbf{E}}' = \frac{1}{\sigma\mu_0}\nabla^2\overline{\mathbf{E}}$$

with $\overline{\mathbf{E}}' = \overline{\mathbf{E}}+\mathbf{U}\times\overline{\mathbf{B}}$ shows how the electric field in the rest frame of a fluid element is related to the second derivative of the electric field (in the lab frame) and the conductivity via a diffusion equation.

- Equations (261)–(262) are nine coupled equations of motion for the components of $\mathbf{U}$, $\overline{\mathbf{B}}$, and $\overline{\mathbf{E}}$. Together with an equation of state [∇p appears in (261)] the system is determined.

4.3.6 *Frozen magnetic field lines*

- If a plasma is at rest, $\mathbf{U} = \mathbf{0}$, eq. (262) reduces also to a simple diffusion equation
$$\partial_t \overline{\mathbf{B}} = \frac{1}{\sigma\mu_0}\nabla^2\overline{\mathbf{B}}. \tag{264}$$
Hence, magnetic fields *diffuse* in a plasma, depending on the conductivity σ of the plasma.

- In fact, if L and τ are the typical length and time scales, respectively, over which $\overline{\mathbf{B}}$ varies, eq. (264) yields
$$\tau = L^2\sigma\mu_0. \tag{265}$$
We see: in the limit of infinite conductivity $\sigma \to \infty$ the magnetic field does (i.e., the magnetic field lines do) not move through the plasma at all, as $\tau \to \infty$. The magnetic field lines only move with the plasma as a whole. This phenomenon is called magnetic *flux freezing*.

- A breakdown of flux freezing occurs during *solar flares*. Let us follow a magnetic field line, starting from the solar photosphere, reaching out into the solar corona, and back to the photosphere. The plasma moving along the photosphere surface carries along the field lines (or, vice versa, the field lines drag along the plasma). Now, as field lines may get more and more intertwined, the magnetic field energy may increase such that it is ultimately released in an erruption of plasma. As field lines start slipping, huge electric fields are created that accelerate the plasma particles.

4.3.7 *Consequences of flux freezing*

- If a plasma cylinder is contracted (or expanded) radially, $R = R(t)$, (e.g., in a z-pinch) we have, because of flux freezing,
$$B(t)\pi R^2(t) = \text{const},$$
and because of mass conservation
$$\rho_m(t)\pi R^2(t) = \text{const}$$
so that
$$\frac{B(t)}{\rho_m(t)} = \text{const} \qquad \text{(perp.)}$$

where 'perp.' refers to contraction/expansion perpendicular to the symmetry axis.

- Now consider a contraction in length $\ell = \ell(t)$ along the symmetry axis. In this case

$$B(t) = \text{const}, \quad \rho_m(t)\ell(t) = \text{const} \quad \Rightarrow \quad \frac{B(t)}{\rho_m(t)\ell(t)} = \text{const} \qquad \text{(para.)}.$$

- Finally, in a spherical contraction $r = r(t)$ we have

$$B(t)r^2(t) = \text{const}, \quad \rho_m(t)r^3(t) = \text{const} \quad \Rightarrow \quad \frac{B(t)}{\rho_m^{2/3}(t)} = \text{const} \qquad \text{(sph.)}.$$

E Why is $Br^2 = \text{const}$?

- The three results can be summarized in

$$\frac{B(t)}{\rho_m^\alpha} = \text{const}, \qquad \alpha = 1, 0, \frac{2}{3} \tag{266}$$

for perpendicular, parallel, or spherical contraction/expansion, respectively.

4.3.8 *Stellar collapse*

- Flux freezing in a spherically symmetric setup should be applicable to stelar collapses.[12]

- First, the collapse of a star to a white dwarf, $r_{\text{star}} = 10^9$ m, $r_{\text{white}} = 10^7$ m, i.e.

$$f_{\rho_m} = 10^6, \qquad f_B = 10^4,$$

with f_{ρ_m} or f_B the factor by which the mass density or the magnetic field changes, respectively. Hence, if the initial magnetic field of the star was 10^{-2} T (a typical value) the white-dwarf field will be 10^2 T, which is observed.

- If the star collapses to a neutron star of $r_{\text{neutron}} = 10^4$ m, we get $f_B = 10^{10}$ and thus $B_{\text{neutron}} = 10^8$ T, which is also observed.

[12] The following examples are taken from KULSRUD's book.

- Equation (265),

$$\tau = L^2 \sigma \mu_0$$

allows us to estimate whether the flux-freezing assumption makes sense to apply to a stellar collapse. To that end we need an estimate for the conductivity. A simple expression for the resistivity[13] reads (Spitzer)

$$\frac{1}{\sigma} = \eta = \frac{\pi e^2 m^{1/2}}{(4\pi\epsilon_0)^2 (k_\mathrm{B} T)^{3/2}} \ln \Lambda, \tag{267}$$

with $\ln \Lambda$ the Coulomb *logarithm*. The latter is only very weakly dependent on the plasma parameters. In fact, it changes only by a factor of 2 as density and temperature vary over 12 and 6 orders of magnitude, respectively. Using

$$\eta[\Omega\mathrm{m}] \simeq 10^{-3} \left(k_\mathrm{B} T[\mathrm{eV}]\right)^{-3/2}$$

we obtain

$$\tau[\mathrm{s}] = 1.3 \times 10^{-3} \left(k_\mathrm{B} T[\mathrm{eV}]\right)^{3/2} (L[\mathrm{m}])^2.$$

- Consider in the white-dwarf case $k_\mathrm{B} T = 10^2\,\mathrm{eV}$ ($T \simeq 10^6$ K), $r_\mathrm{white} = 10^7\,\mathrm{m}$, which leads to $\tau \simeq 10^{-3+2\cdot 3/2+7\cdot 2}\,\mathrm{s} = 10^{14}\,\mathrm{s} = 3 \times 10^6\,\mathrm{y}$. Hence, flux freezing is applicable, as the collapse time is much less than $3 \times 10^6\,\mathrm{y}$.

E The collapse to a neutron star typically occurs on the time scale of seconds. Is the flux-freezing assumption satisfied there as well?

- We considered flux freezing during the collapse of a star to a white dwarf or a neutron star, as in these cases we begin with a plasma in the first place. If initially neutral, non-ionized matter collapses gravitationally to form a star, the situation is different and the MHD equations not sufficient. An estimate for τ would give huge diffusion times and thus would suggest that flux freezing should be valid. An increase of the magnetic field by $f_B = 10^{16}$ (starting from $B_\mathrm{interstellar} = 10^{-10}\,\mathrm{T}$) would be predicted when a protostar collapses to a star.

4.3.9 *Solar wind interaction with the earth's magnetosphere*

- Both solar wind and the earth's magnetosphere are magnetized plasmas.

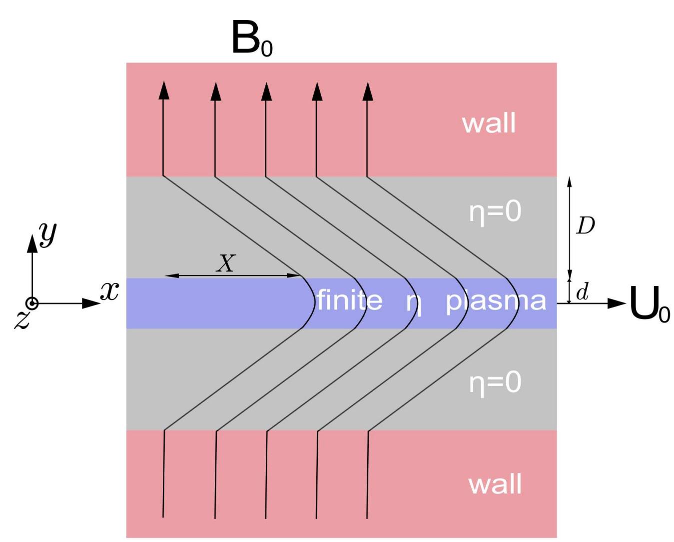
[13] To be derived later, in the context of collisions.

E Why does flux freezing explain that our magnetosphere shields the earth from the solar wind plasma?

- The solar wind deformes the magnetosphere on the sun-facing side [magnetic pressure $\sim B^2$, (255)].
- On the "downstream" side an elongated cavity is formed.
- However, observations have shown that flux freezing is not strictly valid. There is *magnetic reconnection*, allowing the solar wind's magnetic field lines to attach and detach from the magnetosphere field lines.

4.3.10 *Model for magnetic field line slipping*

- We saw above that for infinite conductivity $\sigma \to \infty$ (i.e., vanishing resistivity $\eta = 0$) the magnetic field lines are carried along with the plasma, at speed $\mathbf{U}$. In plasma of zero conductivity (i.e., infinite resistivity) magnetic field lines "slip through" or, changing the reference frame, such a plasma slips through a fixed magnetic field without drag because induction is negligible. At finite resistivity we expect something in between dragging and slipping.
- Consider the following model system:



In the middle between two walls there is a plasma sheet of finite resistivity η and thickness $2d$ (in y direction) streaming with fluid velocity $\mathbf{U}_0$ in

x direction. Between the plasma sheet and the wall (i.e., $d < |y| < d + D$) there is a thin, low-pressure, $\eta = 0$ plasma at rest, $\mathbf{U} = \mathbf{0}$ (in which flux freezing is ideal). The entire system is planar, i.e., translationally invariant in x and z direction, so that all derivatives ∂_x, ∂_z of any quantity must vanish.

- If η was vanishing in the moving plasma sheet as well it would drag along the magnetic field lines forever, giving rise to an infinitely large B_x-component. We are looking for the stationary solution at finite η.
- We expect the transient dynamics up to the point where everything remains stationary to be as follows. When the plasma sheet starts moving it drags magnetic field lines with it until $\mathbf{j}$, due to Ampère's law, increases so much that, in Ohm's law $\eta\mathbf{j}$ becomes comparable to the $\mathbf{U} \times \overline{\mathbf{B}}$-term.
- Because of the symmetry of the problem we have

$$\overline{\mathbf{B}} = B_x\mathbf{e}_x + B_y\mathbf{e}_y, \quad \partial_x B_x = \partial_x B_y = 0 \quad \overset{\nabla\cdot\mathbf{B}=0}{\Rightarrow} \quad \partial_y B_y = 0 \quad \Rightarrow \quad B_y = \text{const} = B_0.$$

- In regions where $\eta = 0$ we have from Ohm's law

$$\overline{\mathbf{E}} = -\mathbf{U} \times \overline{\mathbf{B}} \quad \Rightarrow \quad \overline{\mathbf{E}} = E_z\mathbf{e}_z, \quad E_z = -U_x B_y = 0.$$

- Ampère's (251.5) law yields

$$\mu_0 j_z = -\partial_y B_x.$$

If we assume that the current is neglibly small in the low-density plasma, we have

$$B_x \simeq \text{const} = \mp\frac{X}{D}B_0 \qquad \text{for} \quad d < |y| < d + D,$$

where the upper (lower) sign is for $y > 0$ ($y < 0$).

Inside the plasma sheet where $\eta \neq 0$ we have with Faraday's law (251.4)

$$\partial_y E_z = 0 \qquad \Rightarrow \quad E_z = \text{const}.$$

In order to match this to $E_z = 0$ outside the plasma sheet we have

$$E_z = 0$$

everywhere.

- Hence, the z component of Ohm's law reads

$$(\mathbf{U} \times \overline{\mathbf{B}}) \cdot \mathbf{e}_z = U_x B_y = \eta j_z = -\frac{\eta}{\mu_0} \partial_y B_x .$$

Since B_x drops from XB_0/D at $|y| = d$ to zero at $y = 0$ and U_x increases from 0 to U_0, respectively (while $B_y = B_0 = \text{const}$), we may estimate this as

$$U_0 B_0 \simeq \frac{\eta}{\mu_0} \frac{\frac{X}{D} B_0}{d} .$$

- The "drag distance" thus is

$$X = \frac{U_0 D d \mu_0}{\eta} ,$$

i.e., independent of B_0.

- As U_0 is a velocity,

$$\tau' = \frac{D d \mu_0}{\eta} = D d \sigma \mu_0$$

is the relevant diffusion time, which is of the form (265) if we put $L = \sqrt{Dd}$.

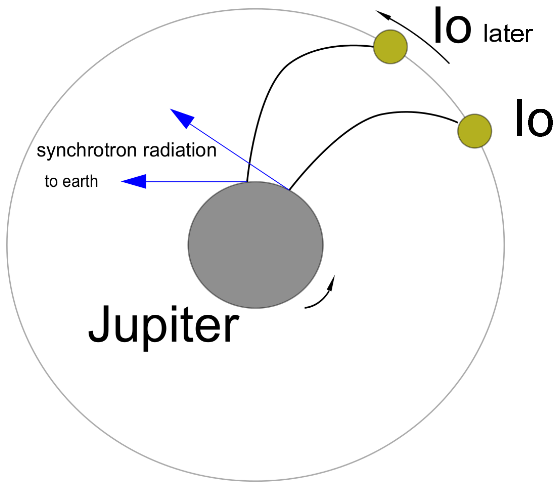
- In the rest frame of the moving plasma sheet we have the electric field

$$\overline{\mathbf{E}}' = \mathbf{U} \times \overline{\mathbf{B}} = \eta j_z = -\frac{\eta}{\mu_0} \partial_y B_x$$

driving the current $\mathbf{j}$ according to Ohm's law.

4.3.11 *The Jupiter-Io connection*

- Synchrotron radiation has been measured from Jupiter. The radiation was found to correlate with the orbiting of the inner moon Io around Jupiter. In fact, the synchrotron radiation is emitted perpendicular to the magnetic field lines connecting Jupiter with Io. The spectrum suggests that the radiation originates from keV electrons, which raises the question how these fast electrons are generated.



- As Jupiter rotates faster (10 hours) than Io revolves (43 hours) the magnetic flux lines would wind up around Jupiter if flux freezing was perfectly valid in both Jupiter's and Io's ionospheres. In fact, in each revolution the magnetic field would increase by 1.2×10^{-3} T.

- Jupiter's ionosphere has a lower conductivity than Io's. Hence the flux lines slip along Jupiter's surface, and thus Jupiter plays the role of the resistive plasma slab in the above model.

- Of course, in the Jupiter-Io connection problem the geometry is very different from the planar setup above. However, we found in our model that a j_z is generated inside the plasma slab. In the Jupiter-Io system the corresponding current is responsible for the emitted synchrotron radiation.

E How?

4.4 WAVES IN PLASMA

- We encountered already Langmuir and ion-acoustic waves. In this section we want to investigate other important examples for waves in a plasma that are accessible to a linearized fluid description.

- Electromagnetic waves are of particular interest because they provide the possibility to interact with and manipulate plasma in actual experiments. In fact, electromagnetic waves are used to compress or heat plasma, and to *diagnose* it.

4.4.1 *Field-free cold plasma*

- If $\overline{\mathbf{E}}_0 = \overline{\mathbf{B}}_0 = \mathbf{u}_{0,\sigma} = \mathbf{0}$ and

$$\overline{\mathbf{E}}_1 = \hat{\mathbf{E}}_1(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad \hat{\mathbf{B}}_1(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{268}$$

$$n_\sigma = n_{0,\sigma} + \hat{n}_{1,\sigma}(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\omega t}, \qquad \hat{\mathbf{u}}_{1,\sigma}(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\omega t}, \qquad \sum_\sigma q_\sigma n_{0,\sigma} = 0, \tag{269}$$

the linearized two-fluid equations (226)–(231) for a cold, collisionless plasma (i.e., $\nabla p_\sigma \simeq \mathbf{0}$) become

$$\begin{aligned}
-\mathrm{i}\omega\hat{n}_{1,\sigma} + n_{0,\sigma}\nabla\cdot\hat{\mathbf{u}}_{1,\sigma} &= 0, & (270)\\
-\mathrm{i}\omega n_{0,\sigma}\hat{\mathbf{u}}_{1,\sigma} &= \frac{q_\sigma n_{0,\sigma}}{m_\sigma}\hat{\mathbf{E}}_1, & (271)\\
\nabla\times\hat{\mathbf{B}}_1 &= \mu_0\sum_\sigma q_\sigma n_{0,\sigma}\hat{\mathbf{u}}_{1,\sigma} - \mathrm{i}\omega\mu_0\epsilon_0\hat{\mathbf{E}}_1, & (272)\\
\epsilon_0\nabla\cdot\hat{\mathbf{E}}_1 &= \sum_\sigma q_\sigma\hat{n}_{1,\sigma}, & (273)\\
\nabla\times\hat{\mathbf{E}}_1 &= \mathrm{i}\omega\hat{\mathbf{B}}_1. & (274)
\end{aligned}$$

- Combining (271) and (272) yields

$$\begin{aligned}
\nabla\times\hat{\mathbf{B}}_1 &= \left(\mu_0\sum_\sigma\frac{q_\sigma^2 n_{0,\sigma}}{-\mathrm{i}\omega m_\sigma} - \mathrm{i}\omega\mu_0\epsilon_0\right)\hat{\mathbf{E}}_1 = -\mathrm{i}\omega\mu_0\epsilon_0\left(1 - \sum_\sigma\frac{q_\sigma^2 n_{0,\sigma}}{\omega^2\epsilon_0 m_\sigma}\right)\hat{\mathbf{E}}_1\\
&= -\mathrm{i}\omega\mu_0\epsilon_0\left(1 - \left[\frac{\omega_\mathrm{p}^2}{\omega^2} + \frac{\omega_\mathrm{p}^2\frac{mn_{0,\mathrm{i}}(Ze)^2}{Me^2 n_{0,\mathrm{e}}}}{\omega^2}\right]\right)\hat{\mathbf{E}}_1\\
&= -\mathrm{i}\omega\mu_0\epsilon_0\underbrace{\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\left[1 + \frac{mZ}{M}\right]\right)}_{\epsilon(\omega)}\hat{\mathbf{E}}_1, \qquad (275)
\end{aligned}$$

where

$$q_\mathrm{e} = -e, \qquad q_\mathrm{i} = Ze, \qquad n_{0,\mathrm{e}} = Zn_{0,\mathrm{i}} \tag{276}$$

with Z the ion charge state.

Hence we find the dielectric function of a field-free plasma to be

$$\boxed{\epsilon(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\left[1 + \frac{mZ}{M}\right] \simeq 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}}\,. \tag{277}$$

E Compare this result with the expression we obtained for the dielectric function $D(\mathbf{k},\omega)$ eq. (193). How do you get from (193) to (277)?

- Taking the curl of FARADAY's law (274) and using $\sqrt{\epsilon_0\mu_0} = 1/c$ we obtain

$$\nabla \times \nabla \times \hat{\mathbf{E}}_1 = \frac{\omega^2}{c^2}\epsilon(\omega)\,\hat{\mathbf{E}}_1, \tag{278}$$

  which is a wave equation for the electric field component of an electromagnetic wave. $\epsilon \to 1$ gives the "usual" electromagnetic waves in vacuum.

- We know that the zeros of the dielectric function tell us the eigenmodes of the system. Here, we find once more a resonance at $\omega \simeq \omega_\mathrm{p}$. The finite ion mass gives a small correction.

- We do not find BOHM-GROSS or the ion-acoustic resonance because of the *cold*-fluid assumption (i.e., no pressure terms). This is valid as long as $\frac{\omega}{k} \gg v_\mathrm{th}$ with k^{-1} the length scale of the plasma perturbation considered.

- With a wave ansatz $\hat{\mathbf{E}}_1(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\hat{\mathbf{E}}$ (278) gives

$$-\mathbf{k} \times \mathbf{k} \times \hat{\mathbf{E}} = \frac{\omega^2}{c^2}\epsilon(\omega)\,\hat{\mathbf{E}}, \tag{279}$$

  which, for

$$\mathbf{k} = k\mathbf{e}_z,$$

  reads

$$\begin{pmatrix} \omega^2 - \omega_\mathrm{p}^2 - k^2c^2 & 0 & 0 \\ 0 & \omega^2 - \omega_\mathrm{p}^2 - k^2c^2 & 0 \\ 0 & 0 & \omega^2 - \omega_\mathrm{p}^2 \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{pmatrix} = \mathbf{0}. \tag{280}$$

- Two of the three solutions to this equation corresponds to electromagnetic waves and will be discussed below.

- The third solution

$$E_x = E_y = 0, \qquad E_z \sim \mathrm{e}^{\mathrm{i}(kz-\omega t)} \quad \Rightarrow \hat{\mathbf{B}}_1 = \mathbf{0}, \qquad \omega^2 = \omega_\mathrm{p}^2$$

  corresponds to the LANGMUIR-wave result for $T_\mathrm{e} = 0$, i.e., only oscillations, no propagation.

Collisional damping

- If we account phenomenologically for collisional damping by introducing a collision frequency ν into (271) for the electrons,

$$-\mathrm{i}\omega\hat{\mathbf{u}}_{1,\mathrm{e}} = -\frac{e}{m}\hat{\mathbf{E}}_1 - \nu\hat{\mathbf{u}}_{1,\mathrm{e}} \qquad \Rightarrow \qquad \hat{\mathbf{u}}_{1,\mathrm{e}} = \frac{e}{\mathrm{i}m(\omega+\mathrm{i}\nu)}\hat{\mathbf{E}}_1$$

we obtain instead of (275)

$$\nabla\times\hat{\mathbf{B}}_1 \;=\; \left(\mu_0\frac{\mathrm{i}e^2 n_{0,\mathrm{e}}}{(\omega+\mathrm{i}\nu)m} - \mathrm{i}\omega\mu_0\epsilon_0\right)\hat{\mathbf{E}}_1 = -\mathrm{i}\omega\mu_0\epsilon_0\left(1-\frac{\omega_\mathrm{p}^2}{\omega(\omega+\mathrm{i}\nu)}\right)\hat{\mathbf{E}}_1$$

so that the dielectric function

$$\epsilon(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega+\mathrm{i}\nu)}$$

is complex.

E Show that the eigen oscillations ω determined by $\epsilon(\omega) = 0$ are damped.

Drifting field-free cold plasma

E Consider a cold electron fluid that drifts with the velocity $\mathbf{u}_{0,\mathrm{e}} = u_{0z}\mathbf{e}_z$. The neutralizing ions can be treated as an immobile background. Consider perturbations $\sim \mathrm{e}^{\mathrm{i}(kz-\omega t)}$. Show that the linearized fluid equations yield the expected dispersion relation

$$(\omega - ku_{0z})^2 = \omega_\mathrm{p}^2. \tag{281}$$

Why is this expected?

- Note that a non-propagating plasma oscillation in the average rest frame of the drifting electrons corresponds to a propagating plasma wave in the lab frame.

4.4.2 *Electromagnetic waves in cold plasma*

- The other two solutions of (280) correspond to electromagnetic waves where $\hat{\mathbf{E}}_1$ is perpendicular to $\mathbf{k}$ and $\hat{\mathbf{B}}_1$, i.e., to electromagnetic waves in a dielectric medium.

- The dispersion relation for these electromagnetic waves in a cold plasma reads

$$\boxed{\omega^2 = k^2c^2 + \omega_\mathrm{p}^2}\,. \tag{282}$$

E Draw a plot ω/ω_p vs kc/ω_p.

E Show that for the group velocity in a cold plasma $v_g = \partial_k\omega < c$ holds but for the phase velocity $v_\varphi = \omega/k > c$.

E What is the energy flux $\mathbf{S}$ of such a wave, i.e., how much field energy is transported per time through a unit area element?

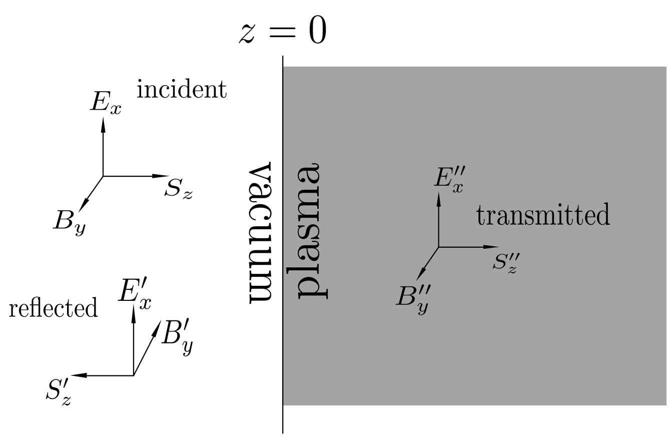
- For $\omega < \omega_\mathrm{p}$ we have

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_\mathrm{p}^2} = \frac{\mathrm{i}}{c}\sqrt{|\omega^2 - \omega_\mathrm{p}^2|}. \tag{283}$$

Hence, fields cannot propagate. Instead, they are damped $\sim \mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{-|k|z}$. This fact can be employed for diagnostic purposes.

4.4.3 *Electromagnetic wave impinging on a plasma boundary*

- Consider the following setup where a plane electromagnetic wave impinges from the vacuum side perpendicularly[14] onto a plasma boundary:



[14] The case of oblique incidence can be also treated but is more cumbersome. The electric field may (in so-called "p-polarization") or may not (in "s polarization") have a component perpendicular to the vacuum-plasma surface.

- Choosing
$$E_x = E_{0x}\,\mathrm{e}^{\mathrm{i}k_0 z}, \qquad k_0^2 c^2 = \omega^2, \qquad \Rightarrow B_y = \frac{E_{0x}}{c}\,\mathrm{e}^{\mathrm{i}k_0 z}$$
and making an ansatz for the transmitted and reflected waves yields
$$E_x' = RE_{0x}\,\mathrm{e}^{-\mathrm{i}k_0 z}, \qquad B_y' = R\frac{E_{0x}}{c}\,\mathrm{e}^{-\mathrm{i}k_0 z},$$
$$E_x'' = TE_{0x}\,\mathrm{e}^{\mathrm{i}k_\mathrm{p} z}, \qquad B_y'' = T\frac{E_{0x}}{\omega/k_\mathrm{p}}\,\mathrm{e}^{\mathrm{i}k_\mathrm{p} z}, \qquad k_\mathrm{p}^2 c^2 = \omega^2 - \omega_\mathrm{p}^2,$$
where
$$R = \frac{k_0 - k_\mathrm{p}}{k_0 + k_\mathrm{p}}, \qquad T = \frac{2k_0}{k_0 + k_\mathrm{p}} \tag{284}$$
are reflection and transmission coefficient, respectively.

- The reflection and transmission coefficients are determined by Maxwell's equations, which require the tangential component of the electric field to be continuous and the tangential component of the magnetic field to jump by the surface current density.

- For $\omega \gg \omega_\mathrm{p}$ the plasma is called highly *underdense* and transparent, $k \simeq k_\mathrm{p}$, $R = 0$, $T = 1$.

- For $\omega \ll \omega_\mathrm{p}$ the plasma is highly *overdense*, $k_\mathrm{p} \to \mathrm{i}\infty$, $T = 0$, $R = -1$. The plasma boundary acts like a perfect mirror.

- For $\omega \ll \omega_\mathrm{p}$ eq. (283) gives
$$k = \mathrm{i}\frac{\omega_\mathrm{p}}{c}$$
which means that the evanescent wave decays over a length
$$\boxed{\delta_\mathrm{s} = \frac{c}{\omega_\mathrm{p}}}\,, \tag{285}$$
which is called the *collisionless skin depth*. The interior of a plasma is thus shielded from radiation of frequency $\omega \ll \omega_\mathrm{p}$.

- For $\omega = \omega_\mathrm{p}$ we have $k_\mathrm{p} = 0$ and thus $R = 1$, $T = 2$.

E The so-called *critical density* n_c is defined as the electron density for which $\omega = \omega_\mathrm{p}$. Calculate the critical density for (a) 800-nm laser light, (b) 2-mm to 3-cm microwave radiation. Which laser wavelengths are needed to shine through a piece of aluminum? Estimate the skin depth of 800-nm laser light in aluminum.

4.4.4 *Field-free warm plasma*

E Show with the help of the linearized two-fluid equations that for cold ions but electrons obeying an equation of state $p/n^\gamma =$ const the dielectric function

$$\epsilon(\omega,k) = 1 - \frac{\omega_{\mathrm{p,i}}^2}{\omega^2} - \frac{\omega_{\mathrm{p,e}}^2}{(\omega^2 - k^2\gamma k_{\mathrm{B}} T_{\mathrm{e}}/m)} \tag{286}$$

(with $\omega_{\mathrm{p,i}}$ and $\omega_{\mathrm{p,e}}$ the ion and electron plasma frequency, respectively) results.

E Show that the roots $\epsilon(\omega,k) = 0$ comprise both the BOHM-GROSS dispersion relation (for the 1D adiabatic choice $\gamma = 3$ for the electrons) and the ion-acoustic wave dispersion relation (for the 1D isothermal choice $\gamma = 1$ and neglect of a small term).

- Remember the condition for the fluid description to be valid: $k^2\lambda_{\mathrm{D}}^2 \ll 1$, so that thermal particles do not travel a wavelength or more in one oscillation period.

4.4.5 *Magnetized plasma*

- We study again the linearized two-fluid equations but now allow for

$$\overline{\mathbf{B}}_0 = B_0\,\mathbf{e}_z.$$

- Equations (270)ff are then the same as before apart from (271)

$$\begin{aligned}
-\mathrm{i}\omega\hat{n}_{1,\sigma} + n_{0,\sigma}\nabla\cdot\hat{\mathbf{u}}_{1,\sigma} &= 0, && (287)\\
-\mathrm{i}\omega n_{0,\sigma}\hat{\mathbf{u}}_{1,\sigma} &= \frac{q_\sigma n_{0,\sigma}}{m_\sigma}\left(\hat{\mathbf{E}}_1 + \hat{\mathbf{u}}_{1,\sigma}\times\overline{\mathbf{B}}_0\right), && (288)\\
\nabla\times\hat{\mathbf{B}}_1 &= \mu_0\sum_\sigma q_\sigma n_{0,\sigma}\hat{\mathbf{u}}_{1,\sigma} - \mathrm{i}\omega\mu_0\epsilon_0\hat{\mathbf{E}}_1, && (289)\\
\epsilon_0\nabla\cdot\hat{\mathbf{E}}_1 &= \sum_\sigma q_\sigma\hat{n}_{1,\sigma}, && (290)\\
\nabla\times\hat{\mathbf{E}}_1 &= \mathrm{i}\omega\hat{\mathbf{B}}_1. && (291)
\end{aligned}$$

- Because of the breaking of the isotropy due to the magnetic field in z direction the dielectric function becomes a *dielectric tensor* $\underline{\underline{\epsilon}}(\omega)$.

- Forcing Ampère's law to assume the form

$$\nabla \times \hat{\mathbf{B}}_1 = -\mathrm{i}\omega\mu_0\epsilon_0 \underline{\underline{\epsilon}}(\omega) \cdot \hat{\mathbf{E}}_1 \tag{292}$$

one finds in a straightforward but cumbersome calculation

$$\underline{\underline{\epsilon}}(\omega) = \begin{pmatrix} \epsilon_1 & \mathrm{i}\epsilon_2 & 0 \\ -\mathrm{i}\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{293}$$

with

$$\epsilon_1 = 1 + \frac{\omega_{\mathrm{p,e}}^2}{\omega_{\mathrm{c,e}}^2 - \omega^2} + \frac{\omega_{\mathrm{p,i}}^2}{\omega_{\mathrm{c,i}}^2 - \omega^2}, \tag{294}$$

$$\epsilon_2 = \frac{\omega_{\mathrm{c,e}}}{\omega} \frac{\omega_{\mathrm{p,e}}^2}{\omega_{\mathrm{c,e}}^2 - \omega^2} - \frac{\omega_{\mathrm{c,i}}}{\omega} \frac{\omega_{\mathrm{p,i}}^2}{\omega_{\mathrm{c,i}}^2 - \omega^2}, \tag{295}$$

$$\epsilon_3 = 1 - \frac{\omega_{\mathrm{p,e}}^2}{\omega^2} - \frac{\omega_{\mathrm{p,i}}^2}{\omega^2} \tag{296}$$

where $\omega_{\mathrm{c,e}} = eB_0/m$ and $\omega_{\mathrm{c,p}} = ZeB_0/M$ are the electron and ion cyclotron frequency, respectively.

- There are resonances in ϵ_1 and ϵ_2 when $\omega^2 = \omega_{\mathrm{c,e}}^2$ or $\omega^2 = \omega_{\mathrm{c,i}}^2$ (*cyclotron resonances*) which may be employed to heat the plasma particles.

- The wave equation (279) now reads

$$-\mathbf{k} \times \mathbf{k} \times \hat{\mathbf{E}} = \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \hat{\mathbf{E}}$$

and can be written as

$$-\mathbf{k}(\mathbf{k} \cdot \hat{\mathbf{E}}) + k^2\hat{\mathbf{E}} = \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \hat{\mathbf{E}}$$

or

$$\boxed{\hat{\mathbf{E}} - \frac{1}{k^2}\mathbf{k}(\mathbf{k} \cdot \hat{\mathbf{E}}) = \frac{1}{n^2} \underline{\underline{\epsilon}}(\omega) \cdot \hat{\mathbf{E}}} \tag{297}$$

with

$$n = \frac{kc}{\omega} = \frac{c}{v_\varphi} \tag{298}$$

the *refractive index* of the plasma.

- Let us take, without loss of generality, $\mathbf{k}$ in the yz plane. Let θ be the angle between $\mathbf{k}$ and the z axis,

$$\frac{\mathbf{k}}{k} = \cos\theta\,\mathbf{e}_z + \sin\theta\,\mathbf{e}_y.$$

Then

$$\begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} - (\cos\theta \hat{E}_z + \sin\theta \hat{E}_y) \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} = \frac{1}{n^2} \begin{pmatrix} \epsilon_1 \hat{E}_x + \mathrm{i}\epsilon_2 \hat{E}_y \\ \epsilon_1 \hat{E}_y - \mathrm{i}\epsilon_2 \hat{E}_x \\ \epsilon_3 \hat{E}_z \end{pmatrix}$$

which results in the homogeneous matrix equation

$$\begin{pmatrix} 1-\epsilon_1/n^2 & -\mathrm{i}\epsilon_2/n^2 & 0 \\ \mathrm{i}\epsilon_2/n^2 & \cos^2\theta - \epsilon_1/n^2 & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & \sin^2\theta - \epsilon_3/n^2 \end{pmatrix} \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = \mathbf{0}. \quad (299)$$

E Show that the requirement for the determinant to vanish leads to the APPLETON-HARTREE equation

$$\boxed{\tan^2\theta = -\frac{(1/n^2 - 1/\epsilon_\mathrm{R})(1/n^2 - 1/\epsilon_\mathrm{L})}{(1/n^2 - 1/\epsilon_3)[1/n^2 - \frac{1}{2}(1/\epsilon_\mathrm{R} + 1/\epsilon_\mathrm{L})]}} \quad (300)$$

where $\epsilon_\mathrm{R} = \epsilon_1 + \epsilon_2$ and $\epsilon_\mathrm{L} = \epsilon_1 - \epsilon_2$.

- The APPLETON-HARTREE equation equation can be used, e.g., to calculate the refractive index n for a wave of frequency ω traveling under an angle θ with respect to $\overline{\mathbf{B}}_0$ in a magnetized plasma of electron plasma frequency ω_p, electron cyclotron frequency ω_c etc.

High-frequency waves propagating parallel to $B_0\mathbf{e}_z$

- For $\omega \gg \omega_\mathrm{c,i}$ and $\theta = 0$ eq. (299) becomes

$$\begin{pmatrix} 1-\epsilon_1/n^2 & -\mathrm{i}\epsilon_2/n^2 & 0 \\ \mathrm{i}\epsilon_2/n^2 & 1-\epsilon_1/n^2 & 0 \\ 0 & 0 & -\epsilon_3/n^2 \end{pmatrix} \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = \mathbf{0} \quad (301)$$

with

$$\epsilon_1 = 1 + \frac{\omega_\mathrm{p,e}^2}{\omega_\mathrm{c,e}^2 - \omega^2}, \qquad \epsilon_2 = \frac{\omega_\mathrm{c,e}}{\omega}\frac{\omega_\mathrm{p,e}^2}{\omega_\mathrm{c,e}^2 - \omega^2}, \qquad \epsilon_3 = 1 - \frac{\omega_\mathrm{p,e}^2}{\omega^2}.$$

- For the solutions corresponding to waves propagating parallel to $B_0\mathbf{e}_z$ (where $\hat{E}_z = 0$) the upper left 2×2-subdeterminant must vanish,

$$(1 - \epsilon_1/n^2)^2 - \epsilon_2^2/n^4 = 0.$$

This gives

$$n^2 = \pm\epsilon_2 + \epsilon_1 = \epsilon_{\mathrm{R,L}}$$

where

$$\epsilon_{\mathrm{R,L}} = 1 + \frac{\omega_{\mathrm{p,e}}^2}{\omega_{\mathrm{c,e}}^2 - \omega^2}\left(1 \pm \frac{\omega_{\mathrm{c,e}}}{\omega}\right) = 1 - \frac{\omega_{\mathrm{p,e}}^2}{\omega(\omega \mp \omega_{\mathrm{c,e}})},$$

and thus

$$k_{\mathrm{R,L}}^2 c^2 = \omega^2\left(1 - \frac{\omega_{\mathrm{p,e}}^2}{\omega(\omega \mp \omega_{\mathrm{c,e}})}\right). \tag{302}$$

- With these solutions (301) yields

$$\begin{aligned}\left(1 - \frac{\epsilon_1}{\epsilon_1 \pm \epsilon_2}\right)\hat{E}_x - \mathrm{i}\frac{\epsilon_2}{\epsilon_1 \pm \epsilon_2}\hat{E}_y &= 0,\\ \mathrm{i}\frac{\epsilon_2}{\epsilon_1 \pm \epsilon_2}\hat{E}_x + \left(1 - \frac{\epsilon_1}{\epsilon_1 \pm \epsilon_2}\right)\hat{E}_y &= 0,\end{aligned}$$

i.e.,

$$\hat{E}_y = \pm\mathrm{i}\hat{E}_x.$$

- This justifies our notation using subscripts R, L, indicating right and left circularly polarized electromagnetic waves

$$\boxed{\overline{\mathbf{E}}_{\mathrm{R,L}}(\mathbf{r}, t) = \hat{E}\ \mathrm{e}^{\mathrm{i}(k_{\mathrm{R,L}}z - \omega t)}(\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y)}\,. \tag{303}$$

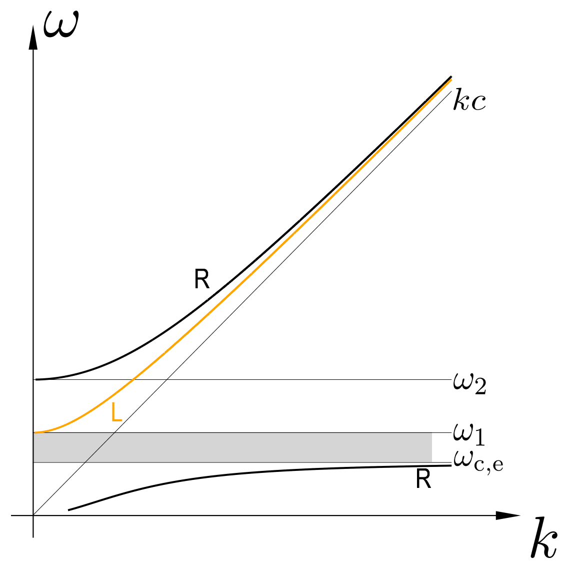
Indeed, for, e.g., a real $\hat{E}$ this is

$$\overline{\mathbf{E}}_{\mathrm{R,L}}(\mathbf{r}, t) = \hat{E}\begin{pmatrix}\cos(\omega t - k_{\mathrm{R,L}}z)\\ \pm\sin(\omega t - k_{\mathrm{R,L}}z)\end{pmatrix},$$

and one can easily convince oneself that the upper (lower) sign describes a right (left) circularly polarized wave (looking in propagation direction).

- The dispersion relation (302) is shown in the following figure.[15]

[15] Only the branches in the first quadrant $\omega, k \geq 0$ are shown.



- The two branches of the right-polarized electromagnetic wave (indicated 'R') approach $\omega_{c,e}$ and kc, respectively, as $k \to \infty$. For $\omega \to 0$ the lower branch is not plotted, as we assumed $\omega \gg \omega_{c,i}$. The upper R-branch goes to

$$\omega_2 = \frac{1}{2}\omega_{c,e}\left(\sqrt{1 + 4\omega_p^2/\omega_{c,e}^2} + 1\right) \tag{304}$$

  for $k \to 0$.

- The branch of the left-polarized electromagnetic wave (indicated 'L') approaches also kc for $k \to \infty$ and

$$\omega_1 = \frac{1}{2}\omega_{c,e}\left(\sqrt{1 + 4\omega_p^2/\omega_{c,e}^2} - 1\right) \tag{305}$$

  for $k \to 0$.

- There are no electromagnetic waves parallel to $B_0\mathbf{e}_z$ in the (gray-shaded) band $\omega_{c,e} < \omega < \omega_1$.

- Only left-polarized electromagnetic waves can propagate in the band $\omega_1 < \omega < \omega_2$.

- For $\omega_p \gg \omega_{c,e}$ we have

$$\omega_{1,2} = \omega_p \mp \frac{1}{2}\omega_{c,e} \to \omega_p$$

  and we recover the usual criterion $\omega > \omega_p$ for electromagnetic waves to propagate in plasma.

- For $\omega > \omega_2$ both left and right-polarized electromagnetic waves can propagate. However, they do so with a different speed because of their different dispersion relation. This leads to so-called Faraday *rotation* of the polarization vector of a linearly polarized electromagnetic wave as it travels through a magnetized plasma along $B_0\mathbf{e}_z$.

E Why?

- For

$$\omega_{\mathrm{c,i}} < \omega \ll \omega_{\mathrm{c,e}}$$

only right-polarized electromagnetic waves can propagate and (302) becomes

$$k_{\mathrm{R}} \simeq \frac{\omega_{\mathrm{p}}}{c}\sqrt{\frac{\omega}{\omega_{\mathrm{c,e}}}}, \qquad \omega_{\mathrm{p}} = \omega_{\mathrm{p,e}}$$

so that

$$\omega(k) = \left(\frac{kc}{\omega_{\mathrm{p}}}\right)^2 \omega_{\mathrm{c,e}}, \qquad k = k_{\mathrm{R}}. \tag{306}$$

- Both group and phase velocity go $\sim k \sim \sqrt{\omega}$, which means that higher frequencies travel faster.

- This effect is observed for electromagnetic waves traveling along the magnetic field lines of the earth in the ionosphere, so-called *whistler waves*.

  An initial broad-band pulse of electromagnetic waves may, for instance, be generated by lightning on the earth's northern hemisphere, the frequencies in the whistler mode range propagate along the magnetic field lines in the ionosphere. The high frequencies are detected first by a receiver on the southern hemisphere, the lower frequencies later. Several returns after reflections back and forth in the ionosphere have been measured. Several whistler waves from the same event may travel through different channels ("force tubes") in the ionosphere so that the same event is heard several times on the other hemisphere.

  As ω must be smaller than the lowest $\omega_{\mathrm{c,e}}$ along the way, whistler waves have an upper limit of $\simeq 100\,\mathrm{kHz}$ so that they can partially be converted to the audio range by simple loudspeakers.

Low-frequency waves propagating parallel to $B_0\mathbf{e}_z$

- For low-frequency electromagnetic waves we have to retain the ion-cyclotron frequency. In that case we obtain, instead of (302),

$$n_{\mathrm{R,L}}^2 = \frac{k_{\mathrm{R,L}}^2 c^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{p,e}}^2}{\omega(\omega \mp \omega_{\mathrm{c,e}})} - \frac{\omega_{\mathrm{p,i}}^2}{\omega(\omega \pm \omega_{\mathrm{c,i}})}. \tag{307}$$

E Derive (307).

- For a neutral plasma

$$\pm\omega_{\mathrm{p,e}}^2 \omega_{\mathrm{c,i}} \mp \omega_{\mathrm{p,i}}^2 \omega_{\mathrm{c,e}} = 0$$

and thus

$$n_{\mathrm{R,L}}^2 = 1 - \frac{\omega_{\mathrm{p,e}}^2 + \omega_{\mathrm{p,i}}^2}{(\omega \mp \omega_{\mathrm{c,e}})(\omega \pm \omega_{\mathrm{c,i}})}. \tag{308}$$

- For $\omega \ll \omega_{\mathrm{c,i}}$ we find

$$k_{\mathrm{R,L}}^2 = \frac{\omega^2}{c^2}\left(1 + \frac{\omega_{\mathrm{p,e}}^2 + \omega_{\mathrm{p,i}}^2}{\omega_{\mathrm{c,e}}\omega_{\mathrm{c,i}}}\right) = \frac{\omega^2}{c^2}\left(1 + mM\frac{\frac{e^2 n_0}{\epsilon_0 m} + \frac{Z^2 e^2 n_0/Z}{\epsilon_0 M}}{Ze^2 B_0^2}\right)$$

$$= \frac{\omega^2}{c^2}\left(1 + \frac{n_0(M + Zm)}{\epsilon_0 Z B_0^2}\right) = k^2.$$

- We see that for $\omega \to 0$ there is no difference between left and right circularly polarized electromagnetic wave. Hence, in this limit a linearly polarized wave propagates without Faraday rotation.

- For $Z = 1$, $\rho_m = n_0(M + m)$ we obtain with the Alfvén velocity

$$V_{\mathrm{A}} = \frac{B_0}{\sqrt{\mu_0 \rho_m}} \tag{309}$$

the dispersion relation for Alfvén *waves*

$$\omega = \frac{k}{\sqrt{\frac{1}{c^2} + \frac{\mu_0 \rho_m}{B_0^2}}} = \frac{k V_{\mathrm{A}}}{\sqrt{1 + V_{\mathrm{A}}^2/c^2}}. \tag{310}$$

E Show that in first order $\omega/\omega_{\mathrm{c,i}}$ the relative difference between the Alfvén velocities of left and right polarized waves is $(V_{\mathrm{A,R}} - V_{\mathrm{A,L}})/V_{\mathrm{A,R}} \simeq \omega/\omega_{\mathrm{c,i}}$.

- The velocity of a transverse wave along a string of tension F and linear density μ is $V = \sqrt{\tau/\mu}$. Comparing this with the ALFVÉN velocity (309) we see that B_0^2 plays the role of tension, providing a restoring force, while the plasma mass density, as usual, provides inertia.
- As $\mathbf{B}_1 \perp \overline{\mathbf{B}}_0$ we may think of ALFVÉN waves as transverse disturbances of magnetic field lines that propagate with the ALFVÉN velocity along the field lines. If flux-freezing is valid the perturbed field lines "assume mass" because of the plasma "attached to them".

ALFVÉN *waves in MHD*

- It is instructive to try a derivation of ALFVÉN waves starting from MHD. The full set of one-fluid equations was equivalent to the set of two-fluid equations. However, while deriving the typical set of MHD equations some terms were neglected. Hence, it is not clear whether ALFVÉN waves are correctly captured in MHD.
- For $p = 0$ eq. (261) becomes

$$\rho_m \partial_t \mathbf{U} = \frac{1}{\mu_0}(\nabla \times \overline{\mathbf{B}}) \times \overline{\mathbf{B}}.$$

- Assuming $\nabla \cdot \mathbf{U} = 0$ [16] and infinite conductivity, (262) simplifies to

$$\partial_t \overline{\mathbf{B}} = (\overline{\mathbf{B}} \cdot \nabla)\mathbf{U} - (\mathbf{U} \cdot \nabla)\overline{\mathbf{B}}.$$

- With the linearization ansatz

$$\overline{\mathbf{B}} = \overline{\mathbf{B}}_0 + \overline{\mathbf{B}}_1(\mathbf{r}, t), \qquad \mathbf{U} = \mathbf{U}_1(\mathbf{r}, t),$$

  i.e., $\overline{\mathbf{B}}_0$ is uniform and constant in time, and $\mathbf{U}_0 = \mathbf{0}$, we obtain

$$\rho_m \partial_t \mathbf{U}_1 = \frac{1}{\mu_0}(\nabla \times \overline{\mathbf{B}}_1) \times \overline{\mathbf{B}}_0,$$

$$\partial_t \overline{\mathbf{B}}_1 = (\overline{\mathbf{B}}_0 \cdot \nabla)\mathbf{U}_1.$$

- With $\overline{\mathbf{B}}_0 = B_0 \mathbf{e}_z$ and solutions of the form

$$\overline{\mathbf{B}}_1(\mathbf{r}, t) = B_x(z,t)\,\mathbf{e}_x + B_y(z,t)\,\mathbf{e}_y, \quad \mathbf{U}_1(\mathbf{r}, t) = U_x(z,t)\,\mathbf{e}_x + U_y(z,t)\,\mathbf{e}_y$$

[16] What this means is discussed at the end of this subsection.

we find

$$\rho_m \partial_t U_x(z,t) = \frac{1}{\mu_0}(\nabla \times \overline{\mathbf{B}}_1)_y B_0 = \frac{1}{\mu_0}\partial_z B_x(z,t) B_0,$$
$$\rho_m \partial_t U_y(z,t) = -\frac{1}{\mu_0}(\nabla \times \overline{\mathbf{B}}_1)_x B_0 = \frac{1}{\mu_0}\partial_z B_y(z,t) B_0,$$

and

$$\partial_t B_x(z,t) = B_0 \partial_z U_x(z,t), \qquad \partial_t B_y(z,t) = B_0 \partial_z U_y(z,t).$$

- Taking derivatives we obtain (assuming $\rho_m = \text{const}$)

$$\partial_t^2 U_{x,y}(z,t) = \frac{1}{\rho_m \mu_0}\partial_t \partial_z B_{x,y}(z,t) B_0, \qquad \partial_z \partial_t B_{x,y}(z,t) = B_0 \partial_z^2 U_{x,y}(z,t)$$

so that

$$\left[\partial_t^2 - \frac{B_0^2}{\rho_m \mu_0}\partial_z^2\right] U_{x,y}(z,t) = 0. \tag{311}$$

- The same wave equation is found for the magnetic field,

$$\left[\partial_t^2 - \frac{B_0^2}{\rho_m \mu_0}\partial_z^2\right] B_{x,y}(z,t) = 0. \tag{312}$$

- A solution, e.g.,

$$B_x = \hat{B}\,\sin(kz - \omega t)$$

is a transverse wave, propagating along the magnetic field lines of $\overline{\mathbf{B}}_0$ with a dispersion relation

$$\omega = V_{\mathrm{A}} k, \qquad V_{\mathrm{A}} = \frac{B_0}{\sqrt{\rho_m \mu_0}}. \tag{313}$$

- Hence we find the same ALFVÉN velocity as in the two-fluid treatment above but a dispersion relation that has not the correct vacuum limit $\omega = kc$ for $\rho_m \to 0$.

[E] Why don't we get the same result as in (310) above?

- It is remarkable that waves can propagate at all in an incompressible, perfectly conducting fluid. In fact, without magnetic field $\overline{\mathbf{B}}_0$ there would be no waves. In order to obtain wave-like solutions without magnetic field one must *not* make the incompressibility assumption $\nabla \cdot \mathbf{U} = 0$ because a restoring force (plus inertia) is required to generate waves.

- In general, plasma is not at all incompressible! However, there may be wave-like solutions which do not compress the plasma, e.g., Alfvén waves.

- What has $\nabla \cdot \mathbf{U} = 0$ to do with incompressibility? We say a fluid is incompressible if a fluid element does not change its density as it moves along, i.e., if the material derivative of the mass density vanishes,

$$\frac{\mathrm{D}\rho_m}{\mathrm{D}t} = \frac{\partial \rho_m}{\partial t} + \mathbf{U} \cdot \nabla \rho_m = 0 \qquad \text{(incompr)}. \tag{314}$$

  Using the continuity equation (251.1), this can be written as

$$\frac{\mathrm{D}\rho_m}{\mathrm{D}t} = -\rho_m \nabla \cdot \mathbf{U} = 0 \qquad \Rightarrow \qquad \nabla \cdot \mathbf{U} = 0 \qquad \text{(incompr)}.$$

[E] Derive the ion-acoustic wave equation "in the MHD way". Show that there is no ion-acoustic wave if one sets $\nabla \cdot \mathbf{U} = 0$.

Waves propagating perpendicular to $B_0\mathbf{e}_z$

- In this case $\theta = \pi/2$ in (299),

$$\begin{pmatrix} 1 - \epsilon_1/n^2 & -\mathrm{i}\epsilon_2/n^2 & 0 \\ \mathrm{i}\epsilon_2/n^2 & -\epsilon_1/n^2 & 0 \\ 0 & 0 & 1 - \epsilon_3/n^2 \end{pmatrix} \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = \mathbf{0}. \tag{315}$$

- We may choose $\mathbf{E}$ parallel to $B_0\mathbf{e}_z$ (*ordinary wave*) or perpendicular to $B_0\mathbf{e}_z$ (*extraordinary wave*)

- In the "ordinary case" we see from (315) that $\epsilon_3 = n^2$ and, for **high frequencies** $\omega^2 \gg \omega_{\mathrm{p,i}}^2$, we find the usual

$$k_{\mathrm{o}} = \pm \frac{\omega}{c} \sqrt{1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}},$$

  as if the magnetic field B_0 were not there.

- In the "extraordinary case" the dispersion relation follows (again) from setting the upper left 2×2-determinant in (315) zero,

$$-\epsilon_1 \left(n^2 - \epsilon_1 \right) - \epsilon_2^2 = 0$$

$$\Rightarrow \quad \left(1+\frac{\omega_{\mathrm{p,e}}^2}{\omega_{\mathrm{c,e}}^2-\omega^2}\right)\left(n^2-\left(1+\frac{\omega_{\mathrm{p,e}}^2}{\omega_{\mathrm{c,e}}^2-\omega^2}\right)\right)-\left(\frac{\omega_{\mathrm{c,e}}}{\omega}\frac{\omega_{\mathrm{p,e}}^2}{\omega_{\mathrm{c,e}}^2-\omega^2}\right)^2=0.$$

- Solving this equation for k in $n = kc/\omega$ we find

$$k_{\mathrm{eo}}^2c^2=\frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{\omega^2-\omega_{\mathrm{h}}^2} \tag{316}$$

with (304), (305) again,

$$\omega_{1,2}=\frac{\omega_{\mathrm{c,e}}}{2}\left[\mp 1+\sqrt{1+\frac{4\omega_{\mathrm{p}}^2}{\omega_{\mathrm{c,e}}^2}}\right].$$

- The new resonance frequency

$$\omega_{\mathrm{h}}=\sqrt{\omega_{\mathrm{p}}^2+\omega_{\mathrm{c,e}}^2} \tag{317}$$

appearing here is known as the *upper hybrid frequency*.

E Where are the "band gaps" here, i.e., for which frequencies do no propagating-wave solutions exist?

- We observe that for $\omega \gg \omega_{\mathrm{h}}$ the refractive index goes to unity.

- The upper hybrid resonance $n^2 \gg 1$ occurs when $\omega \lesssim \omega_{\mathrm{h}}$.

E Show that the extraordinary wave is, in general, not a purely transverse wave.

- At **low frequencies** terms with the ion cyclotron frequency can (again) not be neglected. Making use of $\omega \ll \omega_{\mathrm{c,e}}$ one finds

$$n^2=\frac{k^2c^2}{\omega^2}=-\frac{\omega_{\mathrm{c,e}}^2(\omega_{\mathrm{c,i}}^2-\omega^2)(\epsilon_1^2-\epsilon_2^2)}{\omega_{\mathrm{h}}^2\left(\omega^2-\omega_{\mathrm{c,e}}\omega_{\mathrm{c,i}}\frac{\omega_{\mathrm{p}}^2+\omega_{\mathrm{c,e}}\omega_{\mathrm{c,i}}}{\omega_{\mathrm{p}}^2+\omega_{\mathrm{c,e}}^2}\right)}. \tag{318}$$

The frequency $\omega = \sqrt{\omega_{\mathrm{c,e}}\omega_{\mathrm{c,i}}}$ is called the *lower hybrid frequency*.

E Show that for $\omega \ll \omega_{\mathrm{c,i}}$ we have $n^2 \simeq \epsilon_1$ and

$$n^2 = \frac{k^2 c^2}{\omega^2} = 1 + \frac{\omega_{\mathrm{p,i}}^2}{\omega_{\mathrm{c,i}}^2} \tag{319}$$

and thus

$$\omega^2 = \frac{k^2 V_{\mathrm{A}}^2}{(1 + V_{\mathrm{A}}^2/c^2)^2}. \tag{320}$$

- This wave is called *magnetosonic* wave. It propagates $\perp \mathbf{B}_0$ with mostly $\mathbf{E} \parallel \mathbf{k}$. In contrast, the ALFVÉN wave has $\mathbf{k} \parallel \mathbf{B}_0$ and $\mathbf{E} \perp \mathbf{k}, \mathbf{B}_0$.

E We saw that in magnetized plasma band gaps in the frequency domain appear. Electromagnetic waves with frequencies falling into the respective intervals cannot propagate through the magnetized plasma. This is unfortunate in some situations, in others it is even useful. Think of examples.

Some numbers

- The following table (values taken from KRALL & TRIVELPIECE) gives some typical numbers for the relevant frequencies.

	n_0 [m^{-3}]	B_0 [T]	ω_{p} [s^{-1}]	$\omega_{\mathrm{c,e}}$ [s^{-1}]	$\omega_{\mathrm{c,i}}$ [s^{-1}]
Interplanetary	10^6–10^7	5×10^{-9}	5×10^4	700	1/3 (H)
Ionosphere (80 km)	10^9	$1/3 \times 10^{-4}$	2×10^6	5×10^6	10^2 (O^+)
Ionosphere (100 km)	10^{11}	$1/3 \times 10^{-4}$	2×10^7	5×10^6	10^2 (O^+)

E Why is n_0 higher at 100 km than at 80 km?

5 KINETIC DESCRIPTION OF PLASMA II

- In chapter 3 we postponed the treatment of collisions.
- The Vlasov equation takes into accout only the mean fields $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$, as we ignored the right hand side of (133) involving the pair correlation function.
- In the following sections we discuss the treatment of collisions.

5.1 binary coulomb collisions

- First, let us consider an elementary treatment of an *elastic collision* between two charged particles of charges q_1, q_2 and masses m_1, m_2, respectively.
- Nonrelativistically,[1] the classical equations of motion read

$$\begin{aligned} m_1 \frac{\mathrm{d}^2 \mathbf{r}_1}{\mathrm{d}t^2} &= \frac{q_1 q_2 (\mathbf{r}_1 - \mathbf{r}_2)}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|^3}, \\ m_2 \frac{\mathrm{d}^2 \mathbf{r}_2}{\mathrm{d}t^2} &= \frac{q_1 q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|^3}, \end{aligned}$$

which, when expressed in terms of the center-of-mass and relative coordinates

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{321}$$

turn into

$$\mu \frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} = \frac{q_1 q_2 \mathbf{r}}{4\pi\epsilon_0 |\mathbf{r}|^3}, \tag{322}$$

$$\frac{\mathrm{d}^2 \mathbf{R}}{\mathrm{d}t^2} = \mathbf{0} \tag{323}$$

with

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{324}$$

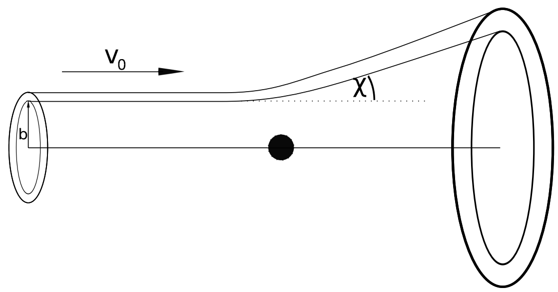
the *reduced mass*.

[1] [E] How would you define a meaningful relativistic center of mass?

- Equation (322) is the equation of motion for a particle of mass μ (and charge, say q_1) that scatters at a *fixed* particle (with charge q_2) centered at the origin $\mathbf{r} = \mathbf{0}$.

- In terms of the *impact parameter b* and the incident velocity v_0 the *asymptotic scattering angle* χ fulfills the relation[2]

$$\tan\frac{\chi}{2} = \frac{q_1 q_2}{4\pi\epsilon_0 \mu v_0^2 b} = \frac{q_1 q_2}{8\pi\epsilon_0 E_{\text{kin}} b}, \qquad E_{\text{kin}} = \frac{1}{2}\mu v_0^2. \tag{325}$$



- The flux scattered into a solid-angle ring $d\Omega'$

$$d\sigma = \frac{d\sigma}{d\Omega'}\, d\Omega' = \frac{d\sigma}{d\Omega'}\, 2\pi \sin\chi \, d\chi$$

originates from an incoming flux $2\pi b\, db$, hence

$$2\pi b\, db = \frac{d\sigma}{d\Omega'}\, 2\pi \sin\chi\, d\chi \qquad \Rightarrow \qquad \frac{d\sigma}{d\Omega'} = \frac{b}{\sin\chi}\frac{db}{d\chi},$$

and using (325), we find the RUTHERFORD *cross section*

$$\frac{d\sigma}{d\Omega'} = \left(\frac{q_1 q_2}{8\pi\epsilon_0 \mu v_0^2 \sin^2\chi/2}\right)^2 = \left(\frac{q_1 q_2}{16\pi\epsilon_0 E_{\text{kin}} \sin^2\chi/2}\right)^2. \tag{326}$$

- It is of interest to calculate the momentum transfer due to binary, elastic COULOMB collisions. The incoming particle is deflected by the angle χ, which means that a portion $1 - \cos\chi$ of the initial momentum is missing along the incident direction after the scattering event. We thus calculate

$$Q = \int_{\chi_{\min}}^{\pi} (1 - \cos\chi) \left(\frac{q_1 q_2}{8\pi\epsilon_0 \mu v_0^2 \sin^2\chi/2}\right)^2 2\pi \sin\chi\, d\chi$$

[2] See any textbook on Classical Mechanics.

$$
\begin{aligned}
&= \int_{\chi_{\min}}^{\pi} 2\sin^2\chi/2 \left(\frac{q_1 q_2}{8\pi\epsilon_0 \mu v_0^2 \sin^2\chi/2}\right)^2 2\pi 2\sin\chi/2\cos\chi/2\,\mathrm{d}\chi \\
&= 8\pi\left(\frac{q_1 q_2}{8\pi\epsilon_0 \mu v_0^2}\right)^2 \int_{\chi_{\min}}^{\pi} \frac{\cos\chi/2}{\sin\chi/2}\,\mathrm{d}\chi \\
&= -8\pi\left(\frac{q_1 q_2}{8\pi\epsilon_0 \mu v_0^2}\right)^2 2\ln\frac{1}{\sin\chi/2}\bigg|_{\chi_{\min}}^{\pi} \\
&= 4\pi\left(\frac{q_1 q_2}{4\pi\epsilon_0 \mu v_0^2}\right)^2 \ln\frac{1}{\sin\chi_{\min}/2} \\
&\simeq 4\pi\left(\frac{q_1 q_2}{4\pi\epsilon_0 \mu v_0^2}\right)^2 \ln\frac{2}{\chi_{\min}}, \qquad \chi_{\min} \ll 1.
\end{aligned}
$$

- We see that the result diverges for $\chi_{\min} \to 0$. This is because of the long-range nature of the COULOMB potential which causes small scattering angles up to infinite impact parameters. In fact, with (325) we have for small $\chi_{\min}$

  $$\frac{2}{\chi_{\min}} = \frac{4\pi\epsilon_0 \mu v_0^2 b_{\max}}{q_1 q_2}$$

  so that

  $$Q \simeq 4\pi\left(\frac{q_1 q_2}{4\pi\epsilon_0 \mu v_0^2}\right)^2 \ln\left(\frac{4\pi\epsilon_0 \mu v_0^2 b_{\max}}{q_1 q_2}\right). \tag{327}$$

- The logarithm in (327) is called COULOMB *logarithm*, usually abbreviated $\ln\Lambda$.

- In a plasma, the particles will not experience a pure COULOMB potential because of DEBYE screening, discussed in section 1.1. Hence we expect that $b_{\max}$ is of order $\mathcal{O}(\lambda_{\mathrm{D}})$.

- If we estimate for electron-electron scattering or electrons scattering on singly charged ions (so that $|q_1 q_2| = e^2$)

  $$\mu v_0^2 \simeq m v_0^2 \simeq 3k_{\mathrm{B}}T$$

  we obtain

  $$\ln\Lambda \simeq \ln\left(\frac{4\pi\epsilon_0 3k_{\mathrm{B}}T b_{\max}}{e^2}\right), \qquad b_{\max} \simeq \lambda_{\mathrm{D}},$$

$$= \ln\left(12\pi n\lambda_{\mathrm{D}}^3\right)$$
$$= \ln\left(\frac{12\pi 3N_{\mathrm{D}}}{4\pi}\right) = \ln\left(9N_{\mathrm{D}}\right) \tag{328}$$

where we used the plasma parameter introduced in eq. (19). As the plasma parameter equals the number of particles in the DEBYE sphere it is typically 10^3–10^5. It enters the calculation of the momentum transfer cross section only logarithmically so that it does not matter much if one chooses for $b_{\max}$ the DEBYE length or several times the DEBYE length.

- A collision frequency ν is obtained by calculating ($\mu \simeq m$)

$$\nu = nv_0Q = 4\pi\left(\frac{e^2}{4\pi\epsilon_0 m}\right)^2\frac{n}{v_0^3}\ln\Lambda.$$

- For a MAXWELL distribution we have

$$\langle v^3\rangle = 8\sqrt{\frac{2}{\pi}}\left(\frac{k_{\mathrm{B}}T}{m}\right)^{3/2}.$$

Ignoring the slow dependence of the COULOMB logarithm on v_0 we may use this average value for v_0^3, in that way obtaining

$$\nu = nv_0Q = \left(\frac{\pi}{2}\right)^{3/2}\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{ne^4}{m^{1/2}(k_{\mathrm{B}}T)^{3/2}}\ln\Lambda. \tag{329}$$

- Apart from the prefactor,[3] this expression leads to the SPITZER resistivity (267) using $\nu = ne^2\eta/m$.

- Note the characteristic $T^{-3/2}$-dependence of the collision frequency. The hotter the plasma the less frequent are binary collisions.

- The ratio of collision frequency to plasma frequency is

$$\frac{\nu}{\omega_{\mathrm{p}}} = \left(\frac{\pi}{2}\right)^{3/2}\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{ne^4\ln\Lambda}{m^{1/2}(k_{\mathrm{B}}T)^{3/2}}\left(\frac{\epsilon_0 m}{e^2 n}\right)^{1/2} = \frac{\ln\Lambda}{32(2\pi)^{1/2}\lambda_{\mathrm{D}}^3 n}. \tag{330}$$

- We see (again) that as long as many particles are in the DEBYE sphere (and $\ln\Lambda$ is not too large) $\nu/\omega_{\mathrm{p}} \ll 1$, and the plasma is in good approximation *collisionless*.

[3] The prefactor depends on the averaging procedure. However, because of our approximations in calculating the COULOMB logarithm we are anyway within logarithmic accuracy only.

5.2 HIERARCHY OF CHARACTERISTIC TIME SCALES

- There are various ways to construct quantities with the dimension of time. One option is[4]
$$\tau_2 \sim \frac{\lambda_\mathrm{D}}{\langle v \rangle} \sim \frac{1}{\omega_\mathrm{p}} \tag{331}$$
where $\langle v \rangle$ is again a typical electron velocity.

- Another time is
$$\tau_1 \sim \frac{1}{\nu} \sim \frac{l}{\langle v \rangle} \tag{332}$$
with l the mean free path. We know from (330) that
$$\tau_2 \ll \tau_1$$
if the plasma is collisionless, i.e., ideal.

- Plasma fluid quantities change on a time scale
$$\tau_0 \sim \frac{L}{C_s} \sim \frac{L}{\langle v \rangle} \tag{333}$$
with L the relevant length (e.g., the wavelength of a plasma wave) and C_s the wave velocity, e.g., the sound speed, which is proportional to $\langle v \rangle$.

- As in ideal plasmas $L \gg l \gg \lambda_\mathrm{D}$ we have
$$\boxed{\tau_2 \ll \tau_1 \ll \tau_0}\,. \tag{334}$$
This hierarchy of the relevant time scales is named after Bogoliubov.

- These times tell us how fast plasma quantities are expected to *relax*. One-particle distribution functions relax on the collisional time scale τ_1, fluid quantities relax no faster than on the hydrodynamic time scale τ_0, while *correlations*, i.e., the correlation function $G(1,2)$ and higher-order correlation functions, are expected to *decay* on the correlation time scale τ_2 and faster.

[4] We are not interested in prefactors of $\mathcal{O}(1)$ here, therefore $\sim$.

5.3 DIELECTRIC PROPAGATOR

- The goal is to derive an explicit expression for the collision term, i.e., the right hand side of (133). 'Explicit' here means 'in terms of F', so that we have a closed equation for F.

- The first equation of the BBGKY hierarchy (133) in chapter 3 reads (all time arguments suppressed)

$$\left[\partial_t + L(1) - \int \mathrm{d}2\, V(1,2)F(2)\right] F(1) = \int \mathrm{d}2\, V(1,2)G(1,2) \qquad (335)$$

  where[5]

$$\begin{aligned} L(1) &= \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} + \frac{q}{m}\left(\mathbf{E}_0 + \mathbf{v}_1 \times \mathbf{B}_0\right) \cdot \nabla_{\mathbf{v}_1}, \\ V(1,2) &= \frac{nq^2}{4\pi\epsilon_0 m}\left(\nabla_{\mathbf{r}_1} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) \cdot \nabla_{\mathbf{v}_1}. \end{aligned}$$

  $F(1)$ is the one-particle distribution function, $G(1,2)$ is the pair-correlation function. Remember how they are related to the i-particle distribution functions $f_i(1,\ldots,i)$:

$$\begin{aligned} C_1(1) &= F(1) = f_1(1), \\ C_2(1,2) &= G(1,2) = f_2(1,2) - C_1(1)C_1(2) = f_2(1,2) - F(1)F(2), \\ C_3(1,2,3) &= H(1,2,3) = f_3(1,2,3) - F(1)F(2)F(3) \\ &\quad -F(1)G(2,3) - F(2)G(3,1) - F(3)G(1,2), \\ &\vdots \end{aligned}$$

  Here we introduced the three-particle correlation function $H(1,2,3)$.

- From the above hierarchy of time scales we expect that

$$\frac{G(1,2)}{F(1)F(2)} \sim N_{\mathrm{D}}^{-1}$$

  and hope for

$$\frac{H(1,2,3)}{F(1)F(2)F(3)} \sim N_{\mathrm{D}}^{-2}.$$

[5] For brevity, we write $V(1,2)$ for one particle species only.

- If this is true we may cut the BBGKY hierarchy after the second equation by neglecting the terms involving $H(1,2,3)$ (as we previously neglected the term $G(1,2)$ when we derived the Vlasov equation in section 3.2).

- Equation (124), expressed in F and G, neglecting H, and with the use of (335) yields

$$\left[\partial_t + L(1) + L(2) - \int \mathrm{d}3\,[V(1,3) + V(2,3)]F(3)\right] G(1,2) - \int \mathrm{d}3\,V(1,3)F(1)G(2,3) - \int \mathrm{d}3\,V(2,3)F(2)G(3,1) = \frac{1}{n}[V(1,2) + V(2,1)]F(1)F(2), \qquad n = \frac{N}{V}. \tag{336}$$

E Derive (336).

- Equations (335) and (336) together determine the time-evolution of the single-particle distribution function F and the pair correlation function G.

- The term on the right hand side of (336) makes the equation inhomogeneous and thus acts as the source term for correlations to build up. Otherwise, if it were not there, G would remain 0 if it was 0 initially.

- In order to find a solution for G we introduce an auxiliary quantity, namely the propagation operator $U(1,1';t-t')$ from single-particle phase-space point $1'$ at time t' to single-particle phase-space point 1 at time t, and express $G(1,2;t)$ as

$$G(1,2;t) = \int \mathrm{d}1' \int \mathrm{d}2' \int_{-\infty}^{t} \mathrm{d}t'\, G_0(1',2';t')U(1,1';t-t')U(2,2';t-t') \tag{337}$$

with the initial condition

$$U(1,1';0) = \delta(1-1'). \tag{338}$$

The hope is that the equation we will derive for U is easier to solve than (336) directly.

Equation (337) has a transparent mathematical interpretation: the correlations $G_0(1',2';t')$ between particles at $1'$ and $2'$ at any time $t' < t$ are propagated by $U(1,1';t-t')U(2,2';t-t')$ to time t where the particles are at 1 and 2.

- Or, the other way round: the correlation $G(1,2;t)$ is built up from all correlations that propagated from all other points (hence the integrals over $1'$ and $2'$) starting from all possible *previous* times (hence the integral over t' restricted to $t' \leq t$, ensuring *causality*).
- We want to derive an expression for U (and G_0). To that end we plug (337) into (336),

$$\begin{aligned}
&\left[\partial_t + L(1) + L(2) - \int d3\,[V(1,3)+V(2,3)]F(3)\right] \\
&\qquad \times \int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')U(1,1';t-t')U(2,2';t-t') \\
&\qquad - \int d3\, V(1,3)F(1) \int d2' \int d3' \int_{-\infty}^{t} dt'\, G_0(2',3';t')U(2,2';t-t')U(3,3';t-t') \\
&\qquad - \int d3\, V(2,3)F(2) \int d3' \int d1' \int_{-\infty}^{t} dt'\, G_0(3',1';t')U(3,3';t-t')U(1,1';t-t') \\
&= \underbrace{\int d1' \int d2' G_0(1',2';t)U(1,1';0)U(2,2';0)}_{G_0(1,2;t)} \\
&\quad + \underline{\int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')[\partial_t U(1,1';t-t')]U(2,2';t-t')} \\
&\quad + \int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')U(1,1';t-t')\partial_t U(2,2';t-t') \\
&\quad + \underline{\int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')[L(1)U(1,1';t-t')]U(2,2';t-t')} \\
&\quad + \int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')U(1,1';t-t')L(2)U(2,2';t-t') \\
&\quad \underline{- \int d3 \int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')[V(1,3)U(1,1';t-t')]F(3;t)U(2,2';t-t')} \\
&\quad - \int d3 \int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t')U(1,1';t-t')V(2,3)U(2,2';t-t')F(3;t) \\
&\qquad \underline{- \int d3\, V(1,3)F(1) \int d2' \int d3' \int_{-\infty}^{t} dt'\, G_0(2',3';t')U(2,2';t-t')U(3,3';t-t')} \\
&\qquad - \int d3\, V(2,3)F(2) \int d3' \int d1' \int_{-\infty}^{t} dt'\, G_0(3',1';t')U(3,3';t-t')U(1,1';t-t') \\
&\qquad\qquad = \frac{1}{n}[V(1,2)+V(2,1)]F(1)F(2).
\end{aligned}$$

- The underlined terms give

$$\begin{aligned}
&\int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t') \left[\left\{\partial_t + L(1) - \int d3\, F(3)V(1,3)\right\} U(1,1';t-t')\right] U(2,2';t-t') \\
&\qquad - \int d3\, V(1,3)F(1) \int d2' \int d3' \int_{-\infty}^{t} dt'\, G_0(2',3';t')U(2,2';t-t')U(3,3';t-t') \\
&=
\end{aligned}$$

$$\int d1' \int d2' \int_{-\infty}^{t} dt'\, G_0(1',2';t') \left[\left\{\partial_t + L(1) - \int d3\, F(3)V(1,3)\right\} U(1,1';t-t')\right] U(2,2';t-t')$$
$$- \int d2' \int d1' \int_{-\infty}^{t} dt' \underbrace{G_0(2',1';t')}_{G_0(1',2';t')} \int d3\, V(1,3)F(1)U(3,1';t-t')U(2,2';t-t').$$

- We see that if U satisfies

$$\boxed{\begin{aligned}\left[\partial_t + L(1) - \int d2\, V(1,2)F(2)\right] U(1,1';t-t') \\ - \int d2\, V(1,2)F(1)U(2,1';t-t') = 0\end{aligned}} \tag{339}$$

  and

$$G_0(1,2;t) = \frac{1}{n}[V(1,2)+V(2,1)]F(1;t)F(2;t), \tag{340}$$

  eq. (336) is fulfilled.

- Hence we have for (337)

$$\boxed{\begin{aligned}&G(1,2;t) \\ &\quad = \frac{1}{n}\int d1' \int d2' \int_{-\infty}^{t} dt'\, [V(1',2')+V(2',1')] \\ &\qquad \times F(1';t')F(2';t')U(1,1';t-t')U(2,2';t-t')\end{aligned}}, \tag{341}$$

  which is a *formal* solution for the pair correlation function.[6]

- Note that eq. (339) is a linearized Vlasov equation for U because with $F = F_0 + \delta F$ up to $\mathcal{O}(\delta F)$

$$\left[\partial_t + L(1) - \int d2\, V(1,2)\{F_0(2)+\delta F(2;t)\}\right]\{F_0(1)+\delta F(1;t)\}$$
$$= \left[\partial_t + L(1) - \int d2\, V(1,2)F_0(2;t)\right]\delta F(1;t) - \int d2\, V(1,2)\delta F(2;t)F_0(1) = 0,$$

  which looks like (339) if we identify $F_0 \to F$ and $\delta F \to U$.

  Hence, we boiled down our original problem of solving a complicated equation for G to solving yet another Vlasov-like equation for U.

[6] Only 'formal' because in order to evaluate $G(1,2;t)$ according (341) one needs U *and* F. But to propagate $F(1;t)$ one needs $G(1,2;t)$.

- In order to proceed towards an explicit expression for the collision term

$$\left.\frac{\partial F}{\partial t}\right|_{c} = \int \mathrm{d}2\, V(1,2)G(1,2) \tag{342}$$

on the right hand side of (335) we have to make further simplifications.

- For simplicity, let us consider the case without external fields ($\mathbf{E}_0 = \mathbf{B}_0 = \mathbf{0}$) so that

$$L(1) = \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1}.$$

- Moreover, we assume a spatially uniform plasma so that

$$F(1;t) = f(\mathbf{v}_1;t), \qquad \int \mathrm{d}^3 v\, f(\mathbf{v};t) = 1 \tag{343}$$

(and $F(1)$ normalized to the position-space volume V, i.e., $\int \mathrm{d}1\, F(1;t) = V$, as it should).

- According to the BOGOLIUBOV time hierarchy we expect $f(\mathbf{v};t)$ to vary on a slower time scale than $G(1,2;t)$ and $U(1,2;t)$. Therefore we consider F stationary in (339) and obtain

$$\begin{aligned}
0 &= \left[\partial_t + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} - \frac{nq^2}{4\pi\epsilon_0 m} \underbrace{\int \mathrm{d}^3 v_2\, f(\mathbf{v}_2)}_{1} \int \mathrm{d}^3 r_2 \left(\nabla_{\mathbf{r}_1} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) \cdot \nabla_{\mathbf{v}_1}\right] U(1,1';t-t') \\
&\quad - \int \mathrm{d}2\, V(1,2) f(\mathbf{v}_1) U(2,1';t-t') \\
&= \left[\partial_t + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} + \frac{nq^2}{4\pi\epsilon_0 m} \underbrace{\left(\int \mathrm{d}^3 r_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\right)}_{0} \cdot \nabla_{\mathbf{v}_1}\right] U(1,1';t-t') \\
&\quad - \int \mathrm{d}2\, V(1,2) f(\mathbf{v}_1) U(2,1';t-t')
\end{aligned}$$

and thus

$$\boxed{\left[\partial_t + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1}\right] U(1,1';t) - \int \mathrm{d}2\, V(1,2) f(\mathbf{v}_1) U(2,1';t) = 0}\,. \tag{344}$$

- This equation can now be solved for U using the FOURIER and LAPLACE transform technique as introduced in section 3.3.1.

- Remember (176),

$$g_{\mathbf{k}}(\mathbf{v},p) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-pt} \underbrace{\frac{1}{(2\pi)^3} \int \mathrm{d}^3 r\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} g(\mathbf{r},\mathbf{v},t),}_{\text{Fourier-transform } g_{\mathbf{k}}(\mathbf{v},t)} \qquad \mathrm{Re}(p) \geq p_0$$

$$= \frac{1}{(2\pi)^3}\int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}} \underbrace{\int_0^\infty dt\, e^{-pt}\, g(\mathbf{r},\mathbf{v},t)}_{\text{Laplace}-\text{transform } g(\mathbf{r},\mathbf{v},p)},$$

and the inverse

$$g(\mathbf{r},\mathbf{v},t) = \int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}} \int_{p_0-i\infty}^{p_0+i\infty} \frac{dp}{2\pi i}\, e^{pt}\, g_{\mathbf{k}}(\mathbf{v},p).$$

- Since for spatially homogeneous systems $U(1,1';t)$ can only depend on $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_1'$ we introduce the FOURIER-transformed propagator

$$U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t) = \frac{1}{(2\pi)^3}\int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, U(1,1';t), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_1' \tag{345}$$

$$U(1,1';t) = \int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}} U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_1'. \tag{346}$$

- The LAPLACE transform and its inverse read

$$U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p) = \int_0^\infty dt\, e^{-pt}\, U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t), \tag{347}$$

$$U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t) = \int_{p_0-i\infty}^{p_0+i\infty} \frac{dp}{2\pi i}\, e^{pt}\, U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p). \tag{348}$$

- Alternatively, we can write with

$$p = -i\omega$$

$$U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';-i\omega) = \int_0^\infty dt\, e^{i\omega t}\, U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t),$$

$$U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t) = \int_{ip_0-\infty}^{ip_0+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';-i\omega).$$

The analytic continuation of $U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)$ to $\operatorname{Re} p < p_0$ (where $U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)$ may have poles) translates to the analytic continuation of $U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';-i\omega)$ to $\operatorname{Im}\omega < p_0$ (where $U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';-i\omega)$ may have poles).

- In order to ensure causality the contour is to be chosen such[7] that

$$U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t) = 0 \qquad \text{for} \quad t < 0. \tag{349}$$

[7] That is, with semi-circles of radius $\to \infty$ to the right of p_0 in the complex p-plane or above ip_0 in the complex ω-plane.

- For $t = 0$ the initial condition (338) implies

$$\begin{aligned} U_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_1'; t=0) &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 r \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \, \delta(1-1'), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_1', \\ &= \frac{\delta(\mathbf{v}_1 - \mathbf{v}_1')}{(2\pi)^3}. \end{aligned} \tag{350}$$

- The FOURIER-transformed $V(1,2)$ reads

$$\begin{aligned} V_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_2) &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 r \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \, \frac{nq^2}{4\pi\epsilon_0 m} \left(\nabla_{\mathbf{r}} \frac{1}{r} \right) \cdot \nabla_{\mathbf{v}_1}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \\ &= \frac{1}{(2\pi)^3} \frac{nq^2}{4\pi\epsilon_0 m} \int \mathrm{d}^3 r \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \left(\nabla_{\mathbf{r}} \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k' \; \mathrm{e}^{\mathrm{i}\mathbf{r}\cdot\mathbf{k}'} \, \frac{4\pi}{k'^2} \right) \cdot \nabla_{\mathbf{v}_1} \\ &= \frac{4\pi}{(2\pi)^6} \frac{nq^2}{4\pi\epsilon_0 m} \int \mathrm{d}^3 r \int \mathrm{d}^3 k' \; \mathrm{e}^{-\mathrm{i}\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')} \; \frac{\mathrm{i}\mathbf{k}'}{k'^2} \cdot \nabla_{\mathbf{v}_1} \\ &= \frac{1}{(2\pi)^3} \frac{nq^2}{\epsilon_0 m} \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \\ &= \frac{\omega_{\mathrm{p}}^2}{(2\pi)^3} \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1}, \end{aligned} \tag{351}$$

where we have used the FOURIER transform of the COULOMB potential.

E Calculate the FOURIER transform of the COULOMB potential.

- With (351) we can write

$$V(1,2) = \frac{\omega_{\mathrm{p}}^2}{(2\pi)^3} \int \mathrm{d}^3 k \; \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \, \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1}, \tag{352}$$

and the FOURIER-LAPLACE-transformed eq. (344) becomes

$$\begin{aligned} 0 &= \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-pt} \int \mathrm{d}^3 r \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_1')} \Big\{ [\partial_t + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1}] \, U(1,1';t), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_1', \\ &\qquad - \int \mathrm{d}2 \frac{\omega_{\mathrm{p}}^2}{(2\pi)^3} \int \mathrm{d}^3 k' \, \mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \frac{\mathrm{i}\mathbf{k}'}{k'^2} \cdot \nabla_{\mathbf{v}_1} f(\mathbf{v}_1) U(2,1';t) \Big\} \\ &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 r \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_1')} \int_0^\infty \mathrm{d}t \partial_t [\, \mathrm{e}^{-pt} U(1,1';t)] \\ &\qquad - \frac{1}{(2\pi)^3} \int \mathrm{d}^3 r \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_1')} \int_0^\infty \mathrm{d}t [\partial_t \, \mathrm{e}^{-pt}] U(1,1';t) + \text{rest} \\ &= -U_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_1', t=0) + p U_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_1', p) \end{aligned}$$

$$
\begin{aligned}
&\underbrace{+\frac{1}{(2\pi)^3}\mathbf{v}_1\cdot\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt}\int \mathrm{d}^3r\;\nabla_{\mathbf{r}_1}[\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_1')}U(1,1';t)]}_{0,\text{ assuming } U(1,1';t) \text{ falls off rapidly enough as } r\to\infty}\\
&-\frac{1}{(2\pi)^3}\mathbf{v}_1\cdot\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt}\int \mathrm{d}^3r\;[\nabla_{\mathbf{r}_1}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_1')}]U(1,1';t)+\mathrm{rest}'\\
=\;& -U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',t=0)+pU_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',p)+\mathrm{i}\mathbf{v}_1\cdot\mathbf{k}U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)\\
&-\frac{\omega_\mathrm{p}^2}{(2\pi)^6}\int \mathrm{d}^3v_2\int_0^\infty \mathrm{d}t\int \mathrm{d}^3k'\,\mathrm{e}^{-pt}\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot\nabla_{\mathbf{v}_1}f(\mathbf{v}_1)\\
&\qquad\times\int \mathrm{d}^3r_2\int \mathrm{d}^3(r_1-r_1')\;\;\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_1')}\,\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)}U(2,1';t)\\
=\;& -U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',t=0)+(p+\mathrm{i}\mathbf{v}_1\cdot\mathbf{k})U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',p)\\
&-\frac{\omega_\mathrm{p}^2}{(2\pi)^6}\int \mathrm{d}^3v_2\int_0^\infty \mathrm{d}t\int \mathrm{d}^3k'\,\mathrm{e}^{-pt}\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot\nabla_{\mathbf{v}_1}f(\mathbf{v}_1)\\
&\qquad\times\int \mathrm{d}^3r_1\int \mathrm{d}^3(r_2-r_1')\;\;\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_1')}\,\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)}U(2,1';t)\\
=\;& -U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',t=0)+(p+\mathrm{i}\mathbf{v}_1\cdot\mathbf{k})U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',p)\\
&-\frac{\omega_\mathrm{p}^2}{(2\pi)^3}\frac{\mathrm{i}\mathbf{k}}{k^2}\cdot\nabla_{\mathbf{v}_1}f(\mathbf{v}_1)\int \mathrm{d}^3v_2\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-pt}\int \mathrm{d}^3(r_2-r_1')\;\;\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1')}U(2,1';t)\\
=\;& -U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',t=0)+(p+\mathrm{i}\mathbf{v}_1\cdot\mathbf{k})U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1',p)\\
&\quad-\omega_\mathrm{p}^2\frac{\mathrm{i}\mathbf{k}}{k^2}\cdot\nabla_{\mathbf{v}_1}f(\mathbf{v}_1)\int \mathrm{d}^3v_2U_{\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_1';p)
\end{aligned}
$$

so that, with (350),

$$
\begin{aligned}
U_{\mathbf{k}}(\mathbf{v},\mathbf{v}',p)\;\;=\;\;&\frac{\delta(\mathbf{v}-\mathbf{v}')}{(2\pi)^3(p+\mathrm{i}\mathbf{v}\cdot\mathbf{k})}\\
&\qquad+\omega_\mathrm{p}^2\frac{\mathrm{i}\mathbf{k}}{k^2(p+\mathrm{i}\mathbf{v}\cdot\mathbf{k})}\cdot\nabla_{\mathbf{v}}f(\mathbf{v})\int \mathrm{d}^3v_2U_{\mathbf{k}}(\mathbf{v}_2,\mathbf{v}';p)
\end{aligned}
$$

and, with $p=-\mathrm{i}\omega$

$$
\begin{aligned}
&U_{\mathbf{k}}(\mathbf{v},\mathbf{v}',-\mathrm{i}\omega)\\
&\quad=\;\;\frac{\mathrm{i}\delta(\mathbf{v}-\mathbf{v}')}{(2\pi)^3(\omega-\mathbf{v}\cdot\mathbf{k})}-\frac{\omega_\mathrm{p}^2}{k^2}\frac{\mathbf{k}\cdot\nabla_{\mathbf{v}}f(\mathbf{v})}{\omega-\mathbf{v}\cdot\mathbf{k}}\int \mathrm{d}^3v_2\,U_{\mathbf{k}}(\mathbf{v}_2,\mathbf{v}';-\mathrm{i}\omega).
\end{aligned}
\tag{353}
$$

- In order to solve this equation for U we integrate over $\mathbf{v}$,

$$
\int \mathrm{d}^3v\,U_{\mathbf{k}}(\mathbf{v},\mathbf{v}',-\mathrm{i}\omega)=\frac{\mathrm{i}}{(2\pi)^3(\omega-\mathbf{v}'\cdot\mathbf{k})}-\frac{\omega_\mathrm{p}^2}{k^2}\int \mathrm{d}^3v_2\,U_{\mathbf{k}}(\mathbf{v}_2,\mathbf{v}';-\mathrm{i}\omega)\int \mathrm{d}^3v\,\frac{\mathbf{k}\cdot\nabla_{\mathbf{v}}f(\mathbf{v})}{\omega-\mathbf{v}\cdot\mathbf{k}}
$$

$$
\Rightarrow\int \mathrm{d}^3v\,U_{\mathbf{k}}(\mathbf{v},\mathbf{v}',-\mathrm{i}\omega)\underbrace{\left[1+\frac{\omega_\mathrm{p}^2}{k^2}\int \mathrm{d}^3v\,\frac{\mathbf{k}\cdot\nabla_{\mathbf{v}}f(\mathbf{v})}{\omega-\mathbf{v}\cdot\mathbf{k}}\right]}_{D(\mathbf{k},\omega)}=\frac{\mathrm{i}}{(2\pi)^3(\omega-\mathbf{v}'\cdot\mathbf{k})}
$$

with $D(\mathbf{k},\omega)$ the dielectric function (180)[8]

- Hence

$$\int \mathrm{d}^3v\, U_{\mathbf{k}}(\mathbf{v},\mathbf{v}',-\mathrm{i}\omega) = \frac{\mathrm{i}}{(2\pi)^3(\omega-\mathbf{v}'\cdot\mathbf{k})D(\mathbf{k},\omega)} \tag{354}$$

and therefore, with $\mathcal{U}_{\mathbf{k}} = (2\pi)^3 U_{\mathbf{k}}$ and (353)

$$\boxed{\mathcal{U}_{\mathbf{k}}(\mathbf{v},\mathbf{v}',-\mathrm{i}\omega) = \frac{\mathrm{i}\delta(\mathbf{v}-\mathbf{v}')}{\omega-\mathbf{v}\cdot\mathbf{k}} - \frac{\mathrm{i}\frac{\omega_\mathrm{p}^2}{k^2}\mathbf{k}\cdot\nabla_{\mathbf{v}}f(\mathbf{v})}{(\omega-\mathbf{v}\cdot\mathbf{k})(\omega-\mathbf{v}'\cdot\mathbf{k})D(\mathbf{k},\omega)}}\,. \tag{355}$$

This is the so-called *dielectric propagator*. The first term is the free propagator, the second term takes COULOMB interaction in first order[9] into account.

5.4 COLLISION TERMS

- Now we have all the prerequisites to continue with (341),

$$G(1,2;t) = \frac{1}{n}\int \mathrm{d}1' \int \mathrm{d}2' \int_{-\infty}^{t} \mathrm{d}t'\,[V(1',2')+V(2',1')] \times F(1';t')F(2';t')U(1,1';t-t')U(2,2';t-t')\,.$$

- The FOURIER transform reads, with (343), for

$$G_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2,t=0) = G_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2)$$

$$\begin{aligned}
G_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2) &= \frac{1}{n}\frac{1}{(2\pi)^3}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}1'\int \mathrm{d}2' \int_{-\infty}^{0}\mathrm{d}t'[V(1',2')+V(2',1')] \\
&\times f(\mathbf{v}_1')f(\mathbf{v}_2')U(1,1';-t')U(2,2';-t') \\
&= \frac{1}{n}\frac{1}{(2\pi)^3}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}1'\int \mathrm{d}2' \int_{0}^{\infty}\mathrm{d}t[V(1',2')+V(2',1')] \\
&\times f(\mathbf{v}_1')f(\mathbf{v}_2')U(1,1';t)U(2,2';t) \\
&= \frac{1}{n}\frac{\omega_\mathrm{p}^2}{(2\pi)^6}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}1'\int \mathrm{d}2' \int_{0}^{\infty}\mathrm{d}t \\
&\times \int \mathrm{d}^3k'\,\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot[\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')}\nabla_{\mathbf{v}_1'} + \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')}\nabla_{\mathbf{v}_2'}]f(\mathbf{v}_1')f(\mathbf{v}_2')U(1,1';t)U(2,2';t)
\end{aligned}$$

[8] Here written for one species σ only.

[9] 'First order' in the sense $\mathcal{O}(N_\mathrm{D}^{-1})$, i.e., the BBGKY hierarchy is cut after the second equation, in which $H(1,2,3)=0$ is set. Moreover, we assumed that F is slowly varying compared to G and U.

$$
\begin{aligned}
&= \frac{1}{n}\frac{\omega_{\mathrm{p}}^2}{(2\pi)^6}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}^3r_1'\int \mathrm{d}^3v_1'\int \mathrm{d}^3r_2'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t \\
&\times \int \mathrm{d}^3k'\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot[\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')}\nabla_{\mathbf{v}_1'}+\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')}\nabla_{\mathbf{v}_2'}]f(\mathbf{v}_1')f(\mathbf{v}_2') \\
&\times \int \mathrm{d}^3\mathbf{k}''\int \mathrm{d}^3\mathbf{k}'''\,\mathrm{e}^{\mathrm{i}\mathbf{k}''\cdot(\mathbf{r}_1-\mathbf{r}_1')}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}'''\cdot(\mathbf{r}_2-\mathbf{r}_2')}U_{\mathbf{k}''}(\mathbf{v}_1,\mathbf{v}_1';t)U_{-\mathbf{k}'''}(\mathbf{v}_2,\mathbf{v}_2';t) \\
&= \frac{1}{n}\frac{\omega_{\mathrm{p}}^2}{(2\pi)^6}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}^3r_1'\int \mathrm{d}^3v_1'\int \mathrm{d}^3r_2'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t \\
&\times \int \mathrm{d}^3\mathbf{k}''\int \mathrm{d}^3\mathbf{k}'''\int \mathrm{d}^3k'\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot\,\mathrm{e}^{-\mathrm{i}\mathbf{k}''\cdot\mathbf{r}_1'}\,\mathrm{e}^{\mathrm{i}\mathbf{k}'''\cdot\mathbf{r}_2'}[\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')}\nabla_{\mathbf{v}_1'}+\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')}\nabla_{\mathbf{v}_2'}] \\
&\times \mathrm{e}^{\mathrm{i}\mathbf{k}''\cdot\mathbf{r}_1}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}'''\cdot\mathbf{r}_2}f(\mathbf{v}_1')f(\mathbf{v}_2')U_{\mathbf{k}''}(\mathbf{v}_1,\mathbf{v}_1';t)U_{-\mathbf{k}'''}(\mathbf{v}_2,\mathbf{v}_2';t) \\
&= \frac{1}{n}\frac{\omega_{\mathrm{p}}^2}{(2\pi)^6}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}^3r_1'\int \mathrm{d}^3v_1'\int \mathrm{d}^3r_2'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t\int \mathrm{d}^3\mathbf{k}'' \\
&\times \int \mathrm{d}^3\mathbf{k}'''\int \mathrm{d}^3k'\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot[\mathrm{e}^{\mathrm{i}\mathbf{r}_1'\cdot(\mathbf{k}'-\mathbf{k}'')}\,\mathrm{e}^{\mathrm{i}\mathbf{r}_2'\cdot(\mathbf{k}'''-\mathbf{k}')}\nabla_{\mathbf{v}_1'}+\mathrm{e}^{-\mathrm{i}\mathbf{r}_1'\cdot(\mathbf{k}'+\mathbf{k}'')}\,\mathrm{e}^{\mathrm{i}\mathbf{r}_2'\cdot(\mathbf{k}'''+\mathbf{k}')}\nabla_{\mathbf{v}_2'}] \\
&\times \mathrm{e}^{\mathrm{i}\mathbf{k}''\cdot\mathbf{r}_1}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}'''\cdot\mathbf{r}_2}f(\mathbf{v}_1')f(\mathbf{v}_2')U_{\mathbf{k}''}(\mathbf{v}_1,\mathbf{v}_1';t)U_{-\mathbf{k}'''}(\mathbf{v}_2,\mathbf{v}_2';t) \\
&= \frac{\omega_{\mathrm{p}}^2}{n}\int \mathrm{d}^3(r_1-r_2)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\int \mathrm{d}^3v_1'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t \\
&\times \int \mathrm{d}^3k'\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot[\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)}\nabla_{\mathbf{v}_1'}f(\mathbf{v}_1')f(\mathbf{v}_2')U_{\mathbf{k}'}(\mathbf{v}_1,\mathbf{v}_1';t)U_{-\mathbf{k}'}(\mathbf{v}_2,\mathbf{v}_2';t) \\
&\qquad\qquad + \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1-\mathbf{r}_2)}\nabla_{\mathbf{v}_2'}f(\mathbf{v}_1')f(\mathbf{v}_2')U_{-\mathbf{k}'}(\mathbf{v}_1,\mathbf{v}_1';t)U_{\mathbf{k}'}(\mathbf{v}_2,\mathbf{v}_2';t)] \\
&= \frac{\omega_{\mathrm{p}}^2}{n}\int \mathrm{d}^3v_1'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t\int \mathrm{d}^3k'\frac{\mathrm{i}\mathbf{k}'}{k'^2}\cdot\int \mathrm{d}^3(r_1-r_2) \\
&\qquad \times[\mathrm{e}^{\mathrm{i}(\mathbf{k}'-\mathbf{k})\cdot(\mathbf{r}_1-\mathbf{r}_2)}\nabla_{\mathbf{v}_1'}f(\mathbf{v}_1')f(\mathbf{v}_2')U_{\mathbf{k}'}(\mathbf{v}_1,\mathbf{v}_1';t)U_{-\mathbf{k}'}(\mathbf{v}_2,\mathbf{v}_2';t) \\
&\qquad\qquad + \mathrm{e}^{-\mathrm{i}(\mathbf{k}'+\mathbf{k})\cdot(\mathbf{r}_1-\mathbf{r}_2)}\nabla_{\mathbf{v}_2'}f(\mathbf{v}_1')f(\mathbf{v}_2')U_{-\mathbf{k}'}(\mathbf{v}_1,\mathbf{v}_1';t)U_{\mathbf{k}'}(\mathbf{v}_2,\mathbf{v}_2';t)] \\
&= (2\pi)^3\frac{\omega_{\mathrm{p}}^2}{n}\int \mathrm{d}^3v_1'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t\frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'}-\nabla_{\mathbf{v}_2'}]f(\mathbf{v}_1')f(\mathbf{v}_2')U_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t)U_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';t).
\end{aligned}
$$

- Hence, with $\mathcal{G}_{\mathbf{k}} = (2\pi)^3 G_{\mathbf{k}}$ we have

$$
\begin{aligned}
&\mathcal{G}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2) \qquad (356)\\
&= \frac{\omega_{\mathrm{p}}^2}{n}\int \mathrm{d}^3v_1'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t\frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'}-\nabla_{\mathbf{v}_2'}]f(\mathbf{v}_1')f(\mathbf{v}_2')\mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';t)\mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';t).
\end{aligned}
$$

- With the LAPLACE transform (348) this can be written as

$$
\mathcal{G}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2) = \frac{\omega_{\mathrm{p}}^2}{n}\int \mathrm{d}^3v_1'\int \mathrm{d}^3v_2'\int_0^\infty \mathrm{d}t\frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'}-\nabla_{\mathbf{v}_2'}]f(\mathbf{v}_1')f(\mathbf{v}_2')
$$

$$\times \int_{p_0-\mathrm{i}\infty}^{p_0+\mathrm{i}\infty} \frac{\mathrm{d}p}{2\pi\mathrm{i}}\, \mathrm{e}^{pt}\, \mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p) \int_{p_0'-\mathrm{i}\infty}^{p_0'+\mathrm{i}\infty} \frac{\mathrm{d}p'}{2\pi\mathrm{i}}\, \mathrm{e}^{p't}\, \mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';p')$$

$$= \frac{\omega_\mathrm{p}^2}{n} \int \mathrm{d}^3v_1' \int \mathrm{d}^3v_2' \int_0^\infty \mathrm{d}t \frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2')$$

$$\times \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}}\, \mathrm{e}^{(p+p')t}\, \mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)\mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';p')$$

$$= -\frac{\omega_\mathrm{p}^2}{n} \int \mathrm{d}^3v_1' \int \mathrm{d}^3v_2' \frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2')$$

$$\times \iint \frac{\mathrm{d}p\,\mathrm{d}p'}{(2\pi\mathrm{i})^2}\, \frac{\mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)\mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';p')}{p+p'}, \qquad \mathrm{Re}(p+p') < 0.$$

Here we do the same analytic continuation as in section 3.3.1.

- In the complex ω-plane the expression reads

$$\mathcal{G}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2) = \frac{\omega_\mathrm{p}^2}{n} \int \mathrm{d}^3v_1' \int \mathrm{d}^3v_2' \frac{\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2')$$

$$\times \iint \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\, \frac{\mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)\mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';p')}{\omega+\omega'}, \qquad \mathrm{Im}(\omega+\omega') < 0.$$

- The p'-integration along a contour that is closed with a clockwise-oriented semi-circle in the $\mathrm{Re}\, p > 0$-plane contributes only at the residue $p' = -p$ with $-2\pi\mathrm{i}$,

$$\mathcal{G}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2) = \frac{\omega_\mathrm{p}^2}{n} \int \mathrm{d}^3v_1' \int \mathrm{d}^3v_2' \frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2')$$

$$\times \int \frac{\mathrm{d}p}{2\pi\mathrm{i}}\, \mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';p)\mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';-p).$$

- Alternatively, in the complex ω'-plane the contour is closed counter-clockwise in the upper half plane, contributing $2\pi\mathrm{i}$ at the residue $\omega' = -\omega$,

$$\mathcal{G}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_2) = \frac{\omega_\mathrm{p}^2}{n} \int \mathrm{d}^3v_1' \int \mathrm{d}^3v_2' \frac{\mathrm{i}\mathbf{k}}{k^2}\cdot[\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2')$$

$$\times \int \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{U}_{\mathbf{k}}(\mathbf{v}_1,\mathbf{v}_1';-\mathrm{i}\omega)\mathcal{U}_{-\mathbf{k}}(\mathbf{v}_2,\mathbf{v}_2';\mathrm{i}\omega).$$

E Draw the contours for the p' (or ω') integration.

E Show that $\mathcal{U}_{-\mathbf{k}}(\mathbf{v},\mathbf{v}';\mathrm{i}\omega)$ is the complex conjugate of $\mathcal{U}_{\mathbf{k}}(\mathbf{v},\mathbf{v}';-\mathrm{i}\omega)$.

- The collision term (342) can be written as

$$\left.\frac{\partial f(\mathbf{v}_1)}{\partial t}\right|_\mathrm{c} = \int \mathrm{d}2\, V(1,2)G(1,2)$$

$$
\begin{aligned}
&= \int \mathrm{d}^3 v_2 \int \mathrm{d}^3 r_2 \, \frac{\omega_\mathrm{p}^2}{(2\pi)^3} \int \mathrm{d}^3 k \; \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \, \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \int \mathrm{d}^3 k' \; \mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}_1 - \mathbf{r}_2)} G_{\mathbf{k}'}(\mathbf{v}_1, \mathbf{v}_2) \\
&= -\omega_\mathrm{p}^2 \int \mathrm{d}^3 k \, \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \int \mathrm{d}^3 v_2 \, G_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_2) \\
&= -(2\pi)^2 \frac{\omega_\mathrm{p}^4}{n} \int \mathrm{d}^3 k \, \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \int \mathrm{d}^3 v_1' \int \mathrm{d}^3 v_2' \frac{\mathbf{k}}{k^2} \cdot [\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2') \\
&\qquad \times \int \mathrm{d}p \; U_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_1'; p) \int \mathrm{d}^3 v_2 \, U_{-\mathbf{k}}(\mathbf{v}_2, \mathbf{v}_2'; -p) \\
&= -\frac{1}{2\pi} \frac{\omega_\mathrm{p}^4}{n} \int \mathrm{d}^3 k \, \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \int \mathrm{d}^3 v_1' \int \mathrm{d}^3 v_2' \frac{\mathbf{k}}{k^2} \cdot [\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2') \\
&\qquad \times \int \mathrm{d}p \; U_{\mathbf{k}}(\mathbf{v}_1, \mathbf{v}_1'; p) \frac{\mathrm{i}}{(2\pi)^3 (-\mathrm{i}p + \mathbf{v}_2' \cdot \mathbf{k}) D(-\mathbf{k}, -\mathrm{i}p)} \\
&= -\frac{1}{(2\pi)^4} \frac{\omega_\mathrm{p}^4}{n} \int \mathrm{d}^3 k \, \frac{\mathrm{i}\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \int \mathrm{d}^3 v_1' \int \mathrm{d}^3 v_2' \frac{\mathbf{k}}{k^2} \cdot [\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2') \\
&\qquad \times \int \mathrm{d}p \; \left[\frac{\delta(\mathbf{v}_1 - \mathbf{v}_1')}{\mathrm{i}p - \mathbf{v}_1 \cdot \mathbf{k}} - \frac{\frac{\omega_\mathrm{p}^2}{k^2} \mathbf{k} \cdot \nabla_{\mathbf{v}_1} f(\mathbf{v}_1)}{(\mathrm{i}p - \mathbf{v}_1 \cdot \mathbf{k})(\mathrm{i}p - \mathbf{v}_1' \cdot \mathbf{k}) D(\mathbf{k}, \mathrm{i}p)} \right] \\
&\qquad\qquad \times \frac{1}{(\mathrm{i}p - \mathbf{v}_2' \cdot \mathbf{k}) D(-\mathbf{k}, -\mathrm{i}p)} \\
&= \frac{1}{(2\pi)^4} \frac{\omega_\mathrm{p}^4}{n} \int \mathrm{d}^3 k \, \frac{\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}_1} \int \mathrm{d}^3 v_1' \int \mathrm{d}^3 v_2' \frac{\mathbf{k}}{k^2} \cdot [\nabla_{\mathbf{v}_1'} - \nabla_{\mathbf{v}_2'}] f(\mathbf{v}_1') f(\mathbf{v}_2') \\
&\qquad \times \int \mathrm{d}\omega \; \left[\frac{\delta(\mathbf{v}_1 - \mathbf{v}_1')}{\omega - \mathbf{v}_1 \cdot \mathbf{k}} - \frac{\frac{\omega_\mathrm{p}^2}{k^2} \mathbf{k} \cdot \nabla_{\mathbf{v}_1} f(\mathbf{v}_1)}{(\omega - \mathbf{v}_1 \cdot \mathbf{k})(\omega - \mathbf{v}_1' \cdot \mathbf{k}) D(\mathbf{k}, \omega)} \right] \\
&\qquad\qquad \times \frac{1}{(\omega - \mathbf{v}_2' \cdot \mathbf{k}) D(-\mathbf{k}, -\omega)} .
\end{aligned}
$$

- It is possible, although cumbersome,[10] to perform all but the $\mathbf{k}$ and one of the velocity integrals via contour integration.
- The result is the Lenard-Balescu *collision term*

$$
\boxed{\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_\mathrm{c} = \frac{\pi \omega_\mathrm{p}^4}{(2\pi)^3 n} \int \mathrm{d}^3 k \, \frac{\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}} \int \mathrm{d}^3 v' \frac{\mathbf{k}}{k^2} \cdot \{ [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] f(\mathbf{v}) f(\mathbf{v}') \} \, \frac{\delta(\mathbf{k} \cdot \mathbf{v} - \mathbf{k} \cdot \mathbf{v}')}{|D(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2}} . \tag{357}
$$

- One can show that with the Lenard-Balescu collision term equation (335) (without external fields) has the following properties:

[10] See, e.g., Ichimaru, *Basic Principles of Plasma Physics*, Appendix B.

1. $f(t) > 0$ if $f(t') > 0$ for $t' < t$.
2. $\frac{d}{dt} \int d^3v\, f(\mathbf{v}) = 0$ [number conservation].
3. $\frac{d}{dt} \int d^3v\, \mathbf{v} f(\mathbf{v}) = 0$ [momentum conservation].
4. $\frac{d}{dt} \int d^3v\, v^2 f(\mathbf{v}) = 0$ [(mean kinetic) energy conservation].
5. A Maxwell distribution is a stationary solution.
6. Any distribution function $f(t = 0)$ approaches a Maxwell distribution as $t \to \infty$ (Boltzmann H-theorem, see below).

- The Lenard-Balescu collision term reduces to the so-called Landau collision term if the dielectric function $D(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})$ is approximated by 1.

  The Landau collision term diverges logarithmically because the integration over k goes like $\int dk\, \frac{1}{k}$.

E Why does the integration over k goes like $\int dk\, \frac{1}{k}$?

- The presence of the dielectric function in the Lenard-Balescu collision term cures the divergence at small k (i.e., scattering with small momentum transfer as it happens for large impact parameters) because $D \sim k^{-2}$. In other words: the dielectric function accounts properly for screening so that small k are not problematic.

- However, both Landau and Lenard-Balescu collision term diverge because of the high-k behavior of the integrand $\sim \int dk\, \frac{1}{k}$ while the elementary treatment on the basis of the Rutherford cross section in 5.1 had no problem with high ks (i.e., $b_{\text{min}} \to 0$). The reason is that our perturbative approach in powers of $1/N_D$ (that is, the assumption that $G/FF \sim N_D^{-1}$, $H/FFF \sim N_D^{-2}$ etc.) does not capture the "granularity" of the Coulomb potential that is required to cause large deflections involving large momentum transfers.

5.4.1 *Connection with* Coulomb *logarithm*

- The Lenard-Balescu collision term contains the dielectric function. We shall now expand the dielectric function in order to show how the Coulomb logarithm emerges. Remember, we encountered the Coulomb logarithm in section 5.1. There we had no divergence that would have required a lower cut-off b_{min}. However, as we are free to write the Coulomb logarithm in the form $\ln \Lambda = \ln(b_{\text{max}}/b_{\text{min}})$ we will be able to identify the Coulomb logarithm in the course of simplifying the dielectric function in this section. In that way we can build a smoothly connected expression for the collision term without divergences.

- The Lenard-Balescu collision term can be written in the form

$$\boxed{\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} = -\nabla_{\mathbf{v}} \cdot \mathbf{J}(\mathbf{v})} \tag{358}$$

with

$$\boxed{\mathbf{J}(\mathbf{v}) = \int \mathrm{d}^3v' \underline{\underline{Q}}(\mathbf{v},\mathbf{v}') \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] f(\mathbf{v}) f(\mathbf{v}')} \tag{359}$$

and the tensor

$$\boxed{Q_{ij}(\mathbf{v},\mathbf{v}') = -\frac{\pi\omega_{\mathrm{p}}^4}{(2\pi)^3 n} \int \mathrm{d}^3k \, \frac{k_i k_j}{k^4} \, \frac{\delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')}{|D(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}} \, . \tag{360}$$

- Noticing that

$$\begin{aligned} D(\mathbf{k},\mathbf{k}\cdot\mathbf{v}) &= 1 + \frac{{\omega_{\mathrm{p}}}^2}{k^2} \int \mathrm{d}^3v' \, \frac{\mathbf{k}\cdot\nabla_{\mathbf{v}'} f(\mathbf{v}')}{\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}'} \\ &= 1 + \frac{k_{\mathrm{B}}T}{k^2\lambda_{\mathrm{D}}^2 m} \int \mathrm{d}^3v' \, \frac{\mathbf{k}\cdot\nabla_{\mathbf{v}'} f(\mathbf{v}')}{\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')} \\ &= 1 + \frac{\psi(\mathbf{k},\mathbf{k}\cdot\mathbf{v})}{k^2\lambda_{\mathrm{D}}^2} \end{aligned}$$

where the dimensionless function

$$\psi(\mathbf{k},\mathbf{k}\cdot\mathbf{v}) = \underbrace{\frac{k_{\mathrm{B}}T}{m}}_{\sim v_{\mathrm{th}}^2} \int \mathrm{d}^3v' \, \frac{\mathbf{k}\cdot\nabla_{\mathbf{v}'} f(\mathbf{v}')}{\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')}$$

does not depend on k but only on the direction $\mathbf{k}/k$, we have

$$Q_{ij}(\mathbf{v},\mathbf{v}') = -\frac{\pi\omega_{\mathrm{p}}^4}{(2\pi)^3 n} \int \mathrm{d}^3k \, \frac{k_i k_j}{k^4} \, \frac{\delta[\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')]}{\left|1 + \frac{\psi(\mathbf{k},\mathbf{k}\cdot\mathbf{v})}{k^2\lambda_{\mathrm{D}}^2}\right|^2} . \tag{361}$$

- If we choose, e.g., $k_1 = k_x \parallel (\mathbf{v} - \mathbf{v}')$ [and hence $k_2 = k_y$ and $k_3 = k_z$ $\perp (\mathbf{v} - \mathbf{v}')$] we have

$$Q_{ij}(\mathbf{v},\mathbf{v}') = -\frac{\pi\omega_{\mathrm{p}}^4}{(2\pi)^3 n |\mathbf{v} - \mathbf{v}'|} \iiint \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \, \frac{k_i k_j \, \delta(k_1)}{k^4 \left|1 + \frac{\psi(\mathbf{k},\mathbf{k}\cdot\mathbf{v})}{k^2\lambda_{\mathrm{D}}^2}\right|^2}$$

and thus $Q_{ij} = 0$ if $i = 1$ or $j = 1$.

- Using polar coordinates for the other components, $k_2 = k\cos\vartheta$, $k_3 = k\sin\vartheta$ we find for, e.g., Q_{33},

$$\begin{aligned}
Q_{33} &= -\frac{\pi\omega_\mathrm{p}^4}{(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\int_0^{k_\mathrm{max}}\mathrm{d}k\,k\,\frac{k^2}{k^4\left|1+\frac{\psi}{k^2\lambda_\mathrm{D}^2}\right|^2}, \qquad \psi=\psi|_{k_1=0},\\
&= -\frac{\pi\omega_\mathrm{p}^4}{(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\int_0^{k_\mathrm{max}}\mathrm{d}k\,\frac{1}{k\,|1+\alpha^2/k^2|^2}, \qquad \alpha^2=\psi/\lambda_\mathrm{D}^2\\
&= -\frac{\pi\omega_\mathrm{p}^4}{(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\int_0^{k_\mathrm{max}/\alpha}\mathrm{d}(k/\alpha)\,\frac{(k/\alpha)^3}{|(k/\alpha)^2+1|^2}\\
&= -\frac{\pi\omega_\mathrm{p}^4}{(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\frac{1}{2}\left[\frac{1}{(k/\alpha)^2+1}+\ln[(k/\alpha)^2+1]\right]_0^{k_\mathrm{max}/\alpha}\\
&= -\frac{\pi\omega_\mathrm{p}^4}{2(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\left[\frac{1}{(k_\mathrm{max}/\alpha)^2+1}+\ln[(k_\mathrm{max}/\alpha)^2+1]-1\right]\\
&\simeq -\frac{\pi\omega_\mathrm{p}^4}{2(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\ln\left[(k_\mathrm{max}/\alpha)^2\right].
\end{aligned}$$

In the last step we made use of the fact that ψ is of order unity[11] and that $k_\mathrm{max}/\alpha \gg 1$.

With $k_\mathrm{max} = 1/b_\mathrm{min}$ we obtain

$$\begin{aligned}
Q_{33} &\simeq -\frac{\pi\omega_\mathrm{p}^4}{2(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\ln\underbrace{\left(1/b_\mathrm{min}^2\alpha^2\right)}_{\simeq\lambda_\mathrm{D}^2/b_\mathrm{min}^2}\\
&\simeq -\frac{\pi\omega_\mathrm{p}^4}{(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta\ln(\underbrace{\lambda_\mathrm{D}}_{b_\mathrm{max}}/b_\mathrm{min})\\
&\simeq -\frac{\pi\omega_\mathrm{p}^4}{(2\pi)^3 n|\mathbf{v}-\mathbf{v}'|}\underbrace{\int_0^{2\pi}\mathrm{d}\vartheta\sin^2\vartheta}_{\pi}\ln\underbrace{(b_\mathrm{max}/b_\mathrm{min})}_{\Lambda}\\
&= -\frac{\omega_\mathrm{p}^4}{8\pi n|\mathbf{v}-\mathbf{v}'|}\ln\Lambda.
\end{aligned}$$

[E] Show that the general tensor $\underline{\underline{Q}}$ can be written with $\mathbf{g} = \mathbf{v}-\mathbf{v}'$, $g = |\mathbf{v}-\mathbf{v}'|$ in the form

$$Q_{ij}(\mathbf{v},\mathbf{v}') = -\frac{\omega_\mathrm{p}^4}{8\pi n}\ln\Lambda\,\frac{g^2\delta_{ij}-g_ig_j}{g^3} \tag{362}$$

[11] At least we expect this from the ideal-plasma limit.

or

$$\underline{\underline{Q}}(\mathbf{v}, \mathbf{v}') = -\frac{\omega_p^4}{8\pi n} \ln \Lambda \frac{g^2 \underline{\underline{1}} - \mathbf{gg}}{g^3}. \tag{363}$$

This is called the LANDAU form of $\underline{\underline{Q}}$.

- Using (362) in (359) and (358) gives the corresponding collision term.

BOLTZMANN *H-theorem*

- We know from Statistical Physics that a MAXWELLian distribution maximizes the entropy, and we thus expect that, starting with an arbitrary distribution function $f(\mathbf{v}, t = 0)$, the collision term will act in such a way that $f(\mathbf{v}, t)$ approaches a MAXWELLian distribution on the collisional time scale τ_1.

- Consider

$$H = \int d^3v\, f \ln f, \tag{364}$$

i.e., an entity proportional to the negative of the entropy.

- Starting with an f that is a VLASOV metaequilibrium (so that only collisions will change f, cf. section 3.2.1) we have

$$\begin{aligned}
\frac{dH}{dt} &= \int d^3v \frac{\partial}{\partial f}(f \ln f) \left(\frac{\partial f}{\partial t}\right)_c \\
&= \int d^3v\, (\ln f) \left(\frac{\partial f}{\partial t}\right)_c + \underbrace{\int d^3v \left(\frac{\partial f}{\partial t}\right)_c}_{\frac{d}{dt}\int d^3v\, f = 0 \text{ see 2. in list above}} \\
&= -\int d^3v\, (\ln f) \nabla_{\mathbf{v}} \cdot \mathbf{J} = \int d^3v \frac{1}{f}(\nabla_{\mathbf{v}} f) \cdot \mathbf{J} \\
&= \int d^3v \frac{1}{f}(\nabla_{\mathbf{v}} f) \cdot \int d^3v'\, \underline{\underline{Q}}(\mathbf{v}, \mathbf{v}') \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] f f'
\end{aligned}$$

where $f' = f(\mathbf{v}')$.

- Interchanging the integration variables $\mathbf{v}$ and $\mathbf{v}'$ and because $\underline{\underline{Q}}(\mathbf{v}, \mathbf{v}') = \underline{\underline{Q}}(\mathbf{v}', \mathbf{v})$ we find that this is the same as

$$\frac{dH}{dt} = \int d^3v' \frac{1}{f'}(\nabla_{\mathbf{v}'} f') \cdot \int d^3v\, \underline{\underline{Q}}(\mathbf{v}, \mathbf{v}') \cdot [\nabla_{\mathbf{v}'} - \nabla_{\mathbf{v}}] f f'$$

so that, adding the two, we obtain

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{1}{2}\iint \mathrm{d}^3v\,\mathrm{d}^3v' \left(\frac{1}{f}(\nabla_{\mathbf{v}}f) - \frac{1}{f'}(\nabla_{\mathbf{v}'}f')\right)\cdot \underline{\underline{Q}}(\mathbf{v},\mathbf{v}')\cdot[\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}]ff' \\
&= \frac{1}{2}\iint \frac{\mathrm{d}^3v\,\mathrm{d}^3v'}{ff'} \left(f'\nabla_{\mathbf{v}}f - f\nabla_{\mathbf{v}'}f'\right)\cdot \underline{\underline{Q}}(\mathbf{v},\mathbf{v}')\cdot(f'\nabla_{\mathbf{v}}f - f\nabla_{\mathbf{v}'}f') \\
&= \frac{1}{2}\iint \frac{\mathrm{d}^3v\,\mathrm{d}^3v'}{ff'}\,\mathbf{F}\cdot\underline{\underline{Q}}\cdot\mathbf{F} \\
&= -\frac{\omega_{\mathrm{p}}^4}{16\pi n}\ln\Lambda \iint \frac{\mathrm{d}^3v\,\mathrm{d}^3v'}{ff'}\,\mathbf{F}\cdot\frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g}\cdot\mathbf{F} = \underbrace{-\frac{\omega_{\mathrm{p}}^4}{16\pi n}\ln\Lambda \iint \frac{\mathrm{d}^3v\,\mathrm{d}^3v'}{ff'}F_\perp^2}_{\leq 0}
\end{aligned}$$

with $\hat{\mathbf{g}} = \mathbf{g}/g$ and $\mathbf{F}_\perp$ the component of $\mathbf{F}$ perpendicular to $\mathbf{g}$.

- We see: H decreases (i.e., the entropy increases). We know already from Statistical Physics that the minimum (i.e., maximum entropy) is reached for a Maxwellian.

5.4.2 *Connection with* Fokker-Planck *equation*

- Instead of digging into the microscopic details of collisions one may adopt a stochastic viewpoint: there are *fluctuations* on the time scale τ_2 which cause the phase space distribution of a particle to *relax* on a time scale τ_1. The Fokker-Planck equation describes such Brownian motion. Hence there should be a connection between our results for the collision term (involving the pair distribution function and the particle interaction) and a Fokker-Planck equation with corresponding coefficients.

- Noticing that

$$\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}g = \frac{\partial}{\partial v_i}\frac{v_j - v_j'}{|\mathbf{v}-\mathbf{v}'|} = -\frac{(v_j - v_j')(v_i - v_i')}{|\mathbf{v}-\mathbf{v}'|^3} + \frac{\delta_{ij}}{|\mathbf{v}-\mathbf{v}'|} = \frac{g^2\delta_{ij} - g_i g_j}{g^3}$$

we can write (sum convention used) with (362), (359), (358)

$$\begin{aligned}
\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} &= \frac{\omega_{\mathrm{p}}^4}{8\pi n}\ln\Lambda\frac{\partial}{\partial v_i}\int \mathrm{d}^3v'\,\underbrace{\frac{g^2\delta_{ij} - g_i g_j}{g^3}}_{\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}g}\left[\frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_j'}\right]f(\mathbf{v})f(\mathbf{v}') \\
&= \frac{\omega_{\mathrm{p}}^4}{8\pi n}\ln\Lambda\frac{\partial}{\partial v_i}\left\{\left(\frac{\partial}{\partial v_j}f(\mathbf{v})\right)\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\int \mathrm{d}^3v'\,g f(\mathbf{v}') - f(\mathbf{v})\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\int \mathrm{d}^3v'\,g\frac{\partial}{\partial v_j'}f(\mathbf{v}')\right\}
\end{aligned}$$

$$= \frac{\omega_p^4}{8\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \left\{ \left(\frac{\partial}{\partial v_j} f(\mathbf{v}) \right) \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \int d^3v' \, g f(\mathbf{v}') + f(\mathbf{v}) \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \int d^3v' \, \frac{\partial g}{\partial v'_j} f(\mathbf{v}') \right\}$$

$$= \frac{\omega_p^4}{8\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \left\{ \left(\frac{\partial}{\partial v_j} f(\mathbf{v}) \right) \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \int d^3v' \, g f(\mathbf{v}') - f(\mathbf{v}) \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_j} \int d^3v' \, g f(\mathbf{v}') \right\}.$$

In the last step we used that $\frac{\partial g}{\partial v'_j} = -\frac{\partial g}{\partial v_j}$.

- We wish to bring this into Fokker-Planck form

$$\boxed{\frac{\partial f(\mathbf{v})}{\partial t} = -\frac{\partial}{\partial v_i}[A_i(\mathbf{v}) f(\mathbf{v})] + \frac{1}{2} \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} [B_{ij}(\mathbf{v}) f(\mathbf{v})]} \tag{365}$$

  where the first term is the *drift term* and the second is the *diffusion term*.

- Rewriting this in the form

$$\begin{aligned} \frac{\partial f(\mathbf{v})}{\partial t} &= \frac{\partial}{\partial v_i} \left[-A_i f + \frac{1}{2} \left(\frac{\partial B_{ij}}{\partial v_j} f + \frac{\partial f}{\partial v_j} B_{ij} \right) \right] \\ &= \frac{\partial}{\partial v_i} \left[\left(-A_i + \frac{1}{2} \frac{\partial B_{ij}}{\partial v_j} \right) f + \frac{1}{2} \frac{\partial f}{\partial v_j} B_{ij} \right] \end{aligned}$$

  we identify

$$\boxed{B_{ij} = \frac{\omega_p^4}{4\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \int d^3v' \, |\mathbf{v} - \mathbf{v}'| f(\mathbf{v}')} \tag{366}$$

  and

$$\begin{aligned} -A_i + \frac{1}{2} \frac{\partial B_{ij}}{\partial v_j} &= -A_i + \frac{\omega_p^4}{8\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_j} \int d^3v' \, g f(\mathbf{v}') \\ &\overset{!}{=} -\frac{\omega_p^4}{8\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_j} \int d^3v' \, g f(\mathbf{v}') \end{aligned}$$

  so that

$$A_i = \frac{\omega_p^4}{4\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_j} \int d^3v' \, g f(\mathbf{v}').$$

  Because $\frac{\partial}{\partial v_j} \frac{\partial}{\partial v_j} g = \frac{g^2 \delta_{jj} - g_j g_j}{g^3} = (3g^2 - g^2)/g^3 = 2/g$ we have

$$\boxed{A_i = \frac{\omega_p^4}{2\pi n} \ln\Lambda \frac{\partial}{\partial v_i} \int d^3v' \, \frac{f(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|}}\,. \tag{367}$$

- We see: with the coefficients A_i and B_{ij} according (366) and (367) the FOKKER-PLANCK equation is equivalent to equation (342) defining the collision term on the right-hand-side of a BOLTZMANN-like equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \mathbf{a}[f] \cdot \nabla_{\mathbf{v}}\right) f = \left.\frac{\partial f}{\partial t}\right|_{\mathrm{c}}. \tag{368}$$

Here we write $\mathbf{a}[f]$ just to indicate that the acceleration $\mathbf{a}$ involves mean fields that in general depend (e.g., via HARTREE-like potentials or MAXWELL's equations) on the distribution function f.

- As discussed above, if we start with a non-MAXWELLian distribution f the collision term $\left.\frac{\partial f}{\partial t}\right|_{\mathrm{c}}$ on the right hand side will be $\neq 0$, causing f to relax towards a MAXWELLian equilibrium distribution. Ultimately, when the equilibrium is reached $\left.\frac{\partial f}{\partial t}\right|_{\mathrm{c}} = 0$, i.e., the MAXWELLian fulfills a VLASOV equation.

- The collision term on the right hand side of the BOLTZMANN-like equation (368) narrows down the many equilibrium solutions of the VLASOV equation discussed in section 3.2.1 to the realistic, collisional equilibrium distribution—a MAXWELLian.

5.4.3 *Electron-electron and electron-ion collisions*

- Generalized to several particle species the LENARD-BALESCU collision term for species σ reads

$$\begin{aligned}\left.\frac{\partial f_\sigma(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} &= \frac{\pi q_\sigma^2}{(2\pi)^3 \epsilon_0^2 m_\sigma} \sum_{\sigma'} \int \mathrm{d}^3k \, q_{\sigma'}^2 n_{\sigma'} \frac{\mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}} \\ &\quad \times \int \mathrm{d}^3v' \frac{\mathbf{k}}{k^2} \cdot \left\{ \left[\frac{1}{m_\sigma} \nabla_{\mathbf{v}} - \frac{1}{m_{\sigma'}} \nabla_{\mathbf{v}'} \right] f_\sigma(\mathbf{v}) f_{\sigma'}(\mathbf{v}') \right\} \frac{\delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')}{|D(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2}\end{aligned}$$

or

$$\boxed{\begin{aligned}\left.\frac{\partial f_\sigma(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} &= \frac{\pi \omega_{\mathrm{p}_\sigma}^2}{(2\pi)^3 n_\sigma} \sum_{\sigma'} \int \mathrm{d}^3k \, \frac{\omega_{\mathrm{p}_{\sigma'}}^2 \mathbf{k}}{k^2} \cdot \nabla_{\mathbf{v}} \\ &\quad \times \int \mathrm{d}^3v' \frac{\mathbf{k}}{k^2} \cdot \left\{ \left[\frac{m_{\sigma'}}{m_\sigma} \nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'} \right] f_\sigma(\mathbf{v}) f_{\sigma'}(\mathbf{v}') \right\} \frac{\delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')}{|D(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2}\end{aligned}} \tag{369}$$

with the multi-component dielectric function

$$D(\mathbf{k}, \omega) = 1 + \sum_\sigma \frac{\omega_{\mathrm{p}_\sigma}^2}{k^2} \int \mathrm{d}^3v \, \frac{\mathbf{k} \cdot \nabla_{\mathbf{v}} f_\sigma(\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v}}. \tag{370}$$

- For the collision term in LANDAU form we have, accordingly,

$$\boxed{\left.\frac{\partial f_\sigma(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} = \frac{q_\sigma^2 \ln\Lambda}{8\pi\epsilon_0^2 m_\sigma} \sum_{\sigma'} q_{\sigma'}^2 n_{\sigma'} \nabla_{\mathbf{v}} \cdot \int \mathrm{d}^3v' \, \frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g} \cdot \left[\frac{1}{m_\sigma}\nabla_{\mathbf{v}} - \frac{1}{m_{\sigma'}}\nabla_{\mathbf{v}'}\right] f_\sigma(\mathbf{v}) f_{\sigma'}(\mathbf{v}')}. \tag{371}$$

- Consider the case of an electron-ion plasma with only one type of ions. Then the sum over σ' runs over e and i, and we have for the electron and ion distribution functions $f = f_\mathrm{e}$ and $F = f_\mathrm{i}$

$$\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} = \left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ee}} + \left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ei}}, \qquad \left.\frac{\partial F(\mathbf{v})}{\partial t}\right|_{\mathrm{c}} = \left.\frac{\partial F(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ie}} + \left.\frac{\partial F(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ii}},$$

respectively.

- Let us first investigate the *electron-electron* collision term, $m_\mathrm{e} = m$,

$$\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ee}} = \frac{e^4 n \ln\Lambda}{8\pi\epsilon_0^2 m^2} \nabla_{\mathbf{v}} \cdot \int \mathrm{d}^3v' \, \frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g} \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] \, f(\mathbf{v}) f(\mathbf{v}'). \tag{372}$$

E Show explicitly that the mean kinetic energy

$$\mathcal{E} = \int \mathrm{d}^3v \, \frac{mv^2}{2} f(\mathbf{v})$$

is conserved in electron-electron collisions.

S We have with $f = f(\mathbf{v})$, $f' = f(\mathbf{v}')$, upon integration by parts,

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} &= \int \mathrm{d}^3v \, \frac{mv^2}{2} \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c,ee}} \\
&= \frac{e^4 n \ln\Lambda}{8\pi\epsilon_0^2 m^2} \int \mathrm{d}^3v \, \frac{mv^2}{2} \nabla_{\mathbf{v}} \cdot \int \mathrm{d}^3v' \, \frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g} \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] \, ff' \\
&= -\frac{e^4 n \ln\Lambda}{8\pi\epsilon_0^2 m^2} \iint \mathrm{d}^3v \mathrm{d}^3v' \, m\mathbf{v} \cdot \frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g} \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] \, ff'
\end{aligned}$$

or

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{e^4 n \ln\Lambda}{8\pi\epsilon_0^2 m^2} \iint \mathrm{d}^3v \mathrm{d}^3v' \, m\mathbf{v}' \cdot \frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g} \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] \, ff'$$

so that

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\frac{e^4 n \ln\Lambda}{16\pi\epsilon_0^2 m^2} \iint \mathrm{d}^3v \mathrm{d}^3v' \, m\mathbf{g} \cdot \frac{\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}}{g} \cdot [\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}] \, ff' = 0$$

because

$$\mathbf{g} \cdot (\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}) = \mathbf{0}$$

($\underline{\underline{1}} - \hat{\mathbf{g}}\hat{\mathbf{g}}$ is the projector onto the space $\perp$ to $\mathbf{g}$).

- The proofs that particle number and momentum are conserved follow along the same line.
- The results are expected as in each binary collision of *like* particles (electron-electron or ion-ion) particle number, like-particle momentum, and like-particle energy are conserved.[12]

E Show that the right hand side of (372) vanishes for a local MAXWELLian distribution

$$f_{\mathrm{M}}(\mathbf{r},\mathbf{v},t) = \frac{V}{N}\frac{n(\mathbf{r},t)}{[2\pi k_{\mathrm{B}}T(\mathbf{r},t)/m]^{3/2}} \exp\left\{-\frac{m[\mathbf{v}-\mathbf{u}(\mathbf{r},t)]^2}{2k_{\mathrm{B}}T(\mathbf{r},t)}\right\} \tag{373}$$

where $n(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$, and $T(\mathbf{r},t)$ are local density, mean velocity, and temperature, respectively.[13]

- Now we consider *electron-ion* collisions, $m_{\mathrm{i}} = M$,

$$\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ei}} = \frac{Ze^4 n \ln\Lambda}{8\pi\epsilon_0^2 m}\nabla_{\mathbf{v}}\cdot\int \mathrm{d}^3v'\,\frac{\underline{\underline{1}}-\hat{\mathbf{g}}\hat{\mathbf{g}}}{g}\cdot\left[\frac{1}{m}\nabla_{\mathbf{v}}-\frac{1}{M}\nabla_{\mathbf{v}'}\right] f(\mathbf{v})F(\mathbf{v}'), \tag{374}$$

where we used $q_{\mathrm{i}} = -Ze$, $Zn_{\mathrm{i}} = n_{\mathrm{e}} = n$.

E Show that the right hand side of (374) vanishes for MAXWELLians f_{M} and F_{M} that have the same temperature and mean velocity $\mathbf{u} = \mathbf{u}_{\mathrm{e}} = \mathbf{u}_{\mathrm{i}}$.

- Hence, the electron-ion collision term acts towards thermalization of electrons and ions.
- Momentum is not conserved for electrons and ions individually, only the total momentum, as there is a collision term $\left.\frac{\partial f(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ei}}$ acting on the electrons but also $\left.\frac{\partial F(\mathbf{v})}{\partial t}\right|_{\mathrm{c,ie}}$ acting on the ions.
- There is another hierarchy of time-scales here. First, momentum is transferred from the electrons to the ions until the mean velocities of both species are equal.

[12] However, after all the approximations that led to the LANDAU form of the collision term who could be sure that these conservation laws are still fulfilled? Hence the explicit check.

[13] We normalized $f_{\mathrm{M}}(\mathbf{r},\mathbf{v},t)$ such that $\int \mathrm{d}^3v \int \mathrm{d}^3r\, f_{\mathrm{M}}(\mathbf{r},\mathbf{v},t) = V$ although normalization factors do not matter for the fact that the collision term vanishes, of course.

- Because of electron-electron collisions the electrons become MAXWELLIAN on the same time-scale.

E Show that ions assume a MAXWELLIAN distribution on a time scale that is longer by a factor $\sqrt{M/m}$.

- Ultimately, electrons and ions assume the same temperature on a time scale M/m slower than the electron-electron collision time.

5.4.4 *Collisions with neutrals*

- In a weakly ionized plasma there may be more collisions between neutrals and the charged plasma particles (i.e., electrons or ions) than between the charged particles themselves.

- The interaction between a charged plasma particle and a neutral is *short-range*, not COULOMBIC. As a consequence, the interaction time is small compared to the time between the collisions, unlike in collisions between the charged plasma constituents.

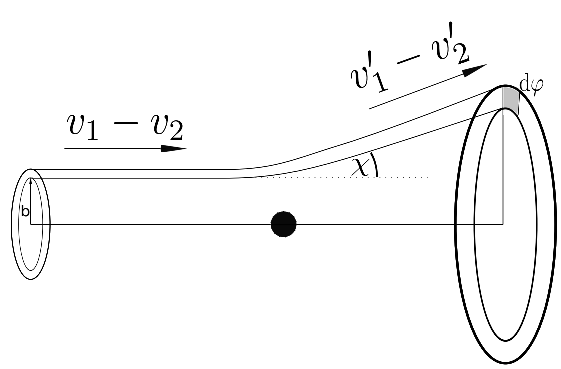
- BOLTZMANN developed a statistical model for low-density, weakly ionized gases and elastic collisions. How does the collision term in the BOLTZMANN equation (368),

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \mathbf{a}\cdot\nabla_{\mathbf{v}}\right) f = \left.\frac{\partial f}{\partial t}\right|_{c}, \tag{375}$$

  look like for such a situation?

- We write, similar to section 5.1,

$$\mathrm{d}\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = b\,\mathrm{d}b\,\mathrm{d}\varphi, \qquad \mathrm{d}\Omega = \sin\chi\,\mathrm{d}\chi\,\mathrm{d}\varphi.$$



- Consider particles with distribution functions $f_{\sigma_1}(\mathbf{v}_1, t)$ and $f_{\sigma_2}(\mathbf{v}_2, t)$.
- The flux of "projectile particles" of species σ_2 into $b\,\mathrm{d}b\,\mathrm{d}\varphi$ is

$$\frac{N_{\sigma_2}}{V} f_{\sigma_2}(\mathbf{v}_2, t)|\mathbf{v}_1 - \mathbf{v}_2|\mathrm{d}^3 v_2 b\,\mathrm{d}b\,\mathrm{d}\varphi,$$

  which has the dimension s^{-1}.
- If we multiply this by the number of available "target particles"[14] per volume

$$\frac{N_{\sigma_1}}{V} f_{\sigma_1}(\mathbf{v}_1, t)\,\mathrm{d}^3 v_1$$

  we obtain

$$\frac{N_{\sigma_2}}{V} f_{\sigma_2}(\mathbf{v}_2, t)|\mathbf{v}_1 - \mathbf{v}_2|\frac{N_{\sigma_1}}{V} f_{\sigma_1}(\mathbf{v}_1, t)\,\underbrace{b\,\mathrm{d}b\,\mathrm{d}\varphi}_{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega}\;\mathrm{d}^3 v_1 \mathrm{d}^3 v_2,$$

  i.e., the number of collisions per time and volume.

BOLTZMANN *Stoßzahlansatz*

- The change of f_{σ_2} because of collisions that scatter particles out of the interval $[\mathbf{v}_2, \mathbf{v}_2 + \mathbf{dv}_2]$ is described by[15]

$$\frac{N_{\sigma_2}}{V}\frac{\partial f_{\sigma_2}(\mathbf{v}_2)}{\partial t}\bigg|_{\mathrm{c,out}} \mathrm{d}^3 v_2 = -\mathrm{d}^3 v_2 \sum_{\sigma_1} \frac{N_{\sigma_2}}{V}\frac{N_{\sigma_1}}{V}\int \mathrm{d}^3 v_1 \int \mathrm{d}\Omega\, f_{\sigma_1}(\mathbf{v}_1, t) f_{\sigma_2}(\mathbf{v}_2, t)|\mathbf{v}_1 - \mathbf{v}_2|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}.$$

- The change of f_{σ_2} because of collisions that scatter particles into the interval $[\mathbf{v}_2, \mathbf{v}_2 + \mathbf{dv}_2]$ is described by

$$\begin{aligned}\frac{N_{\sigma_2}}{V}\frac{\partial f_{\sigma_2}(\mathbf{v}_2)}{\partial t}\bigg|_{\mathrm{c,in}} \mathrm{d}^3 v_2 &= \mathrm{d}^3 v_2' \sum_{\sigma_1} \frac{N_{\sigma_2}}{V}\frac{N_{\sigma_1}}{V}\int \mathrm{d}^3 v_1' \int \mathrm{d}\Omega'\, f_{\sigma_1}(\mathbf{v}_1', t) f_{\sigma_2}(\mathbf{v}_2', t)|\mathbf{v}_1' - \mathbf{v}_2'|\frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'} \\ &= \mathrm{d}^3 v_2 \sum_{\sigma_1} \frac{N_{\sigma_2}}{V}\frac{N_{\sigma_1}}{V}\int \mathrm{d}\Omega \int \mathrm{d}^3 v_1\, f_{\sigma_1}(\mathbf{v}_1', t) f_{\sigma_2}(\mathbf{v}_2', t)|\mathbf{v}_1 - \mathbf{v}_2|\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}.\end{aligned}$$

  In the last step we made use of the symmetry of the elastic scattering process,

$$\frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'}\mathrm{d}\Omega' = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega, \qquad |\mathbf{v}_1' - \mathbf{v}_2'| = |\mathbf{v}_1 - \mathbf{v}_2|, \qquad \mathrm{d}^3 v_1'\,\mathrm{d}^3 v_2' = \mathrm{d}^3 v_1\,\mathrm{d}^3 v_2.$$

[14] One may view the projectiles as targets and vice versa, of course.

[15] We suppress here that the cross section may depend on the species, $\mathrm{d}\sigma_{\sigma_1\sigma_2}/\mathrm{d}\Omega \simeq \mathrm{d}\sigma/\mathrm{d}\Omega$.

- The net change of f_{σ_2} because of collisions thus reads

$$\left.\frac{\partial f_{\sigma_2}(\mathbf{v}_2)}{\partial t}\right|_c = \sum_{\sigma_1}\frac{N_{\sigma_1}}{V}\int d^3v_1\int d\Omega\,[f_{\sigma_1}(\mathbf{v}_1',t)f_{\sigma_2}(\mathbf{v}_2',t) - f_{\sigma_1}(\mathbf{v}_1,t)f_{\sigma_2}(\mathbf{v}_2,t)]|\mathbf{v}_1-\mathbf{v}_2|\frac{d\sigma}{d\Omega},$$

and renaming $\sigma_2 = \sigma$, $\mathbf{v}_2 = \mathbf{v}$, $\mathbf{v}_2' = \mathbf{v}'$, $\sigma_1 = \beta$, we obtain the Boltzmann *collision term*

$$\boxed{\left.\frac{\partial f_\sigma(\mathbf{v})}{\partial t}\right|_c = \sum_\beta\frac{N_\beta}{V}\int d^3v_1\int d\Omega\,\frac{d\sigma}{d\Omega}\,|\mathbf{v}-\mathbf{v}_1|[f_\sigma(\mathbf{v}',t)f_\beta(\mathbf{v}_1',t) - f_\sigma(\mathbf{v},t)f_\beta(\mathbf{v}_1,t)]}\,. \tag{376}$$

- The velocities $\mathbf{v}'$ and $\mathbf{v}_1'$ determine into which solid angle element $d\Omega$ the particles scatter. Hence, the integration over $d\Omega\,\frac{d\sigma}{d\Omega}$ can be formally rewritten as integrals over $\mathbf{v}_1'$ and $\mathbf{v}'$, weighted with an appropriate probability $W(\mathbf{v},\mathbf{v}_1,\mathbf{v}',\mathbf{v}_1')$.

- One can show that with the Rutherford cross section for $\frac{d\sigma}{d\Omega}$ and appropriate cut-offs the Boltzmann equation with the collision term (376) reproduces the Landau form expression (371).

E Show that the Boltzmann collision term (also) conserves the particle number.

E Show that the Boltzmann collision term (also) vanishes for Maxwellians.

E Show that the Boltzmann collision term (also) leads to an increase in entropy.

- The Boltzmann collision term (376) is still quite involved. A drastic simplification is made by introducing a velocity-dependent or even constant *mean free time* $\tau_\sigma(v) \simeq \tau_\sigma = \nu_\sigma^{-1}$ and use

$$\boxed{\left.\frac{\partial f_\sigma}{\partial t}\right|_c = \frac{f_{\sigma 0}(\mathbf{r},\mathbf{v}) - f_\sigma(\mathbf{r},\mathbf{v},t)}{\tau}} \tag{377}$$

(Krook *model*).

- This collision term will cause the distribution function f_σ to relax towards $f_{\sigma 0}$, which should be a local Maxwellian (373)

$$f_{\sigma 0}(\mathbf{r},\mathbf{v},t) = \frac{V}{N_\sigma}\frac{n_\sigma(\mathbf{r},t)}{[2\pi k_B T_\sigma(\mathbf{r},t)/m_\sigma]^{3/2}}\exp\left\{-\frac{m_\sigma[\mathbf{v}-\mathbf{u}_\sigma(\mathbf{r},t)]^2}{2k_B T_\sigma(\mathbf{r},t)}\right\},$$

$$\frac{n_\sigma(\mathbf{r},t)}{N_\sigma/V} = \int \mathrm{d}^3v\, f_\sigma(\mathbf{r},\mathbf{v},t),$$
$$\frac{\mathbf{u}_\sigma(\mathbf{r},t)}{N_\sigma/V} = \frac{1}{n_\sigma(\mathbf{r},t)} \int \mathrm{d}^3v\, \mathbf{v} f_\sigma(\mathbf{r},\mathbf{v},t),$$
$$\frac{3k_\mathrm{B}}{m_\sigma} T_\sigma(\mathbf{r},t) = \frac{N_\sigma}{V}\frac{1}{n_\sigma(\mathbf{r},t)} \int \mathrm{d}^3v\, [\mathbf{v} - \mathbf{u}_\sigma(\mathbf{r},t)]^2.$$

- This choice ensures that upon integrating the BOLTZMANN equation over velocity we obtain

$$\begin{aligned} \partial_t n_\sigma(\mathbf{r},t) + \nabla_\mathbf{r} \cdot [n_\sigma(\mathbf{r},t)\mathbf{u}_\sigma(\mathbf{r},t)] &= \frac{1}{\tau}\frac{N_\sigma}{V} \int \mathrm{d}^3v [f_{\sigma 0}(\mathbf{r},\mathbf{v}) - f_\sigma(\mathbf{r},\mathbf{v},t)] \\ &= \frac{N_\sigma}{V} \frac{n_\sigma(\mathbf{r},t) - n_\sigma(\mathbf{r},t)}{\tau} = 0, \end{aligned}$$

  i.e., continuity, without sources or sinks of particles of type σ.

- Overall momentum is not conserved anymore because the prescribed $f_{\sigma 0}$ will cause f_σ to assume a MAXWELLian centered around the local fluid velocity $\mathbf{u}_\sigma$ without actually caring about where the momentum goes. Consider, e.g., $\mathbf{u}_\sigma = \mathbf{0}$. Then f_σ will ultimately fulfill $\int \mathrm{d}^3v\, \mathbf{v} f_\sigma = \mathbf{0}$, and any initial momentum will be lost because its transfer to the target particles is not taken into account. This is a good approximation if the target particles are heavy, as in the case of neutral atoms as target particles and electrons as charged projectile particles. This is an example of a LORENTZ model, i.e., a model where particles diffuse through immobile target particles.

5.5 COLLISIONAL VS COLLISIONLESS PLASMA

- We had in section 5.1 for collisions between *charged* plasma particles

$$\nu = 4\pi \left(\frac{q_1 q_2}{4\pi\epsilon_0 \mu}\right)^2 \frac{n}{v_0^3} \ln\Lambda, \qquad \ln\Lambda = \ln\frac{b_\mathrm{max}}{b_\mathrm{min}}, \quad b_\mathrm{min} = \frac{|q_1 q_2|}{4\pi\epsilon_0 \mu v_0^2}$$

  with μ the reduced mass and b_min a characteristic length that appears in the argument of the logarithm, and which later assumed the role of a lower cut-off for the LENARD-BALESCU and the LANDAU collision term.

- Introducing a cross section (in section 5.1 called Q)

$$\sigma = \frac{\nu}{nv}$$

we have

$$\sigma = 4\pi \left(\frac{q_1 q_2}{4\pi\epsilon_0 \mu v_0^2} \right)^2 \ln\Lambda = 4\pi b_{\min}^2 \ln\Lambda.$$

E For electron-electron scattering, how is $b_{\min}$ related to the "classical electron radius" and σ to the THOMSON scattering cross section?

- For like particle collisions (ee or ii) we have $\mu v_0^2 \sim \frac{1}{2} m v_{\text{th}}^2 = k_B T$ so that the cross section for ee collisions and singly ionized ii collisions are about the same.
- As $\nu \simeq n v_{\text{th}} \sigma$, and the thermal velocities differ by $\sqrt{M/m}$ for the same temperature,

$$v_{\text{th,e}} = \sqrt{M/m}\, v_{\text{th,i}}$$

we see that for singly ionized ions also

$$\nu_e = \sqrt{M/m}\, \nu_i.$$

Some numbers for astrophysical plasmas

- Consider the electron-electron cross section at the *surface of the sun* where (with $\ln\Lambda \simeq 15$, $\mu = m_e/2$, $v_0 \simeq 2v_{\text{th,e}}$)

$$k_B T \simeq 1\,\text{eV} \quad \Rightarrow \quad v_{\text{th,e}} = 6 \times 10^5\,\text{m/s} \quad \Rightarrow \quad \sigma_{ee} \simeq 4 \times 10^{-16}\,\text{m}^2.$$

The neutral density and the electron and ion density are, respectively,

$$n_{\text{neutral}} = 10^{22}\,\text{m}^{-3}, \qquad n_{e,i} = 10^{18}\,\text{m}^{-3}.$$

The electron and ion density is smaller because only some metals are ionized at the surface of the sun.

The ee-collision frequency and mean free path are

$$\nu_{ee} = n_e v_{\text{th,e}} \sigma_{ee} \simeq 10^7\,\text{s}^{-1} \quad \Rightarrow \quad l_{ee} = \frac{v_{\text{th,e}}}{\nu_{ee}} \simeq 10^{-2}\,\text{m}.$$

- Typical cross sections for collisions of charged plasma particles with neutral atoms are

$$\sigma_{en} \simeq 10^{-20} - 10^{-19}\,\text{m}^2$$

so that

$$\nu_{en} = n_n v_{\text{th,e}} \sigma_{en} \simeq 10^4 n_e v_{\text{th,e}} 10^{-4} \sigma_{ee} = \nu_{ee}$$

and thus also

$$l_{en} \simeq l_{ee} = 10^{-2}\,\text{m}.$$

- At the *center of the sun* the fractional ionization is much higher and collisions with neutrals can be neglected. One has

$$k_B T \simeq 10^3\,\mathrm{eV}, \quad n_e \simeq 10^{32}\,\mathrm{m}^{-3}, \quad \sigma_{ee} \simeq 10^{-23}\,\mathrm{m}^2, \quad \nu_{ee} \simeq 10^{16}\,\mathrm{s}^{-1}, \quad l_{ee} \simeq 10^{-10}\,\mathrm{m}.$$

  The ee cross section is smaller than at the surface but this is overcompensated by the much higher density so that the mean free path decreases by eight orders of magnitude.

- In the *interstellar medium*

$$k_B T \simeq 1\,\mathrm{eV}, \quad n_e \simeq 10^6\,\mathrm{m}^{-3}, \quad \sigma_{ee} \simeq 10^{-16}\,\mathrm{m}^2, \quad \nu_{ee} \simeq 10^{-4}\,\mathrm{s}^{-1}, \quad l_{ee} \simeq 10^9\,\mathrm{m}.$$

- In a *molecular cloud*

$$k_B T \simeq 10\,\mathrm{K} \simeq 10^{-3}\,\mathrm{eV}, \quad \sigma_{ee} \simeq 10^{-11}\,\mathrm{m}^2, \quad n \simeq 10^9\,\mathrm{m}^{-3},$$

  but the fractional ionization is only 10^{-5} so that

$$n_e \simeq 10^4\,\mathrm{m}^{-3} \quad \nu_{ee} \simeq 10^{-3}\,\mathrm{s}^{-1}, \quad l_{ee} \simeq 10^6\,\mathrm{m}.$$

  However, as

$$\sigma_{en} \simeq 10^{-20}\,\mathrm{m}^2 \quad \Rightarrow \quad \nu_{en} \simeq n \underbrace{v_{th,e}}_{10^4\,\mathrm{m/s}} \sigma_{en} \simeq 10^{-9}\,\mathrm{s}^{-1} \quad \Rightarrow \quad l_{en} \simeq 10^{13}\,\mathrm{m}.$$

  we still have

$$l_{en} \gg l_{ee}$$

  despite the low ionization degree. The reason is that the Coulomb cross section σ_{ee} becomes large at low temperatures.

- At high temperatures the Coulomb cross section is small but there are less neutrals. Hence Coulomb collisions also dominate.

- If the length scales L and the time scales T we are interested in are all such that

$$L \gg l, \qquad T \gg \nu^{-1}$$

  the plasma may be treated as *collisional*, and we can apply a fluid description because the plasma is locally in thermodynamic equilibrium and thus the use of some local equation of state is justified.

- If the length scales L and the time scales T we are interested in are all such that

$$L \ll l, \qquad T \ll \nu^{-1}$$

the plasma may be treated as *collisionless*, and we may use the Vlasov equation, neglecting collisions altogether, as the mean field is sufficient. This is usually the case if time scales $T \lesssim \omega_{\mathrm{p}}^{-1}$ are of interest [see remark after (330)].

6 INSTABILITIES

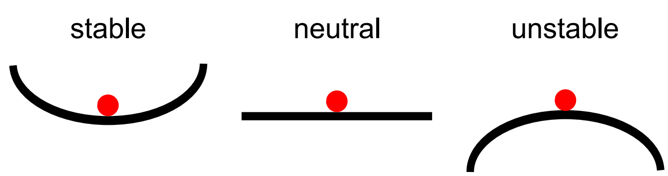
- We have learned in the previous chapter that collisions will bring the plasma to local thermal equilibrium. This happens on time scales that are given by the times between collisions.

- *Collective* dynamics happening on the time-scale ω_{p}^{-1} are faster. In fact, on that time scale collisions can usually be neglected.

- Hence, the question arises what happens to the *metaequilibria* discussed in section 3.2.1 on time scales too short for collisions to achieve local thermal equilibrium.

- It turns out that some of the metaequilibria are stable, i.e., they really last until collisions drive them to the "real" thermal equilibrium.

- However, there are also unstable metaequilibria where the slightest perturbation drives the system away from this metaequilibrium.



- Unstable equilibria in plasma are called *plasma instabilities*.

- Besides their relevance in astrophysics (see below) plasma instabilities are also a vital issue in practical applications such as magnetic or inertial confinement fusion. In both cases one tries to *confine* plasma by *fields*, an endeavour that is plagued by instabilities.

- One of the methods to find out whether an equilibrium is stable or unstable is *normal mode analysis*. One assumes a perturbation and derives a (linearized) equation for the (temporal or spatial) evolution of this perturbation.

- Let us assume that the equation for the perturbation x reads

$$\ddot{x} + 2A\dot{x} + Bx = 0, \qquad A, B \in \mathbb{R}$$

where A and B are some coefficients. The ansatz $x(t) = \mathrm{e}^{-\mathrm{i}\omega t}$ yields

$$\omega^2 + 2\mathrm{i}\omega A - B = 0 \qquad \Rightarrow \qquad \omega_{1,2} = -\mathrm{i}A \pm \sqrt{B - A^2}.$$

We see:

1. If $B < A^2$ the frequency ω is purely imaginary, and the solution is exponential, $x(t) = \mathrm{e}^{\pm Ct}$, $C \in \mathbb{R}$. The "exploding" solutions $\mathrm{e}^{|C|t}$ describe instabilities.
2. For $A = 0$ and $B > 0$ the solution is purely oscillatory.
3. For $A \neq 0$ and $B > A^2$ the solution is oscillatory and exponentially growing (unstable) or damped (stable).

- The theory of plasma instabilities is so rich and the examples of unstable plasma configurations one could study so multifaceted that we have to restrict ourselves to only a few illustrative examples in this lecture.

6.1 TWO-STREAM INSTABILITY

- Consider two uniform, cold, electron beams of densities $n_{\alpha 0}$ and $n_{\beta 0}$ and fluid velocities $u_{\alpha 0}\mathbf{e}_z$ and $u_{\beta 0}\mathbf{e}_z$. The ions are assumed to form an immobile, uniform background that neutralizes the total charge. The whole setup is one-dimensional.

- This situation is a (meta)equilibrium, because there are no fields and no gradients. The fluid equations [without magnetic, pressure, and collision terms, cf. (226), (227)]

$$\begin{aligned} \partial_t n_{\sigma 0} + \nabla \cdot (n_{\sigma 0}\mathbf{u}_{\sigma 0}) &= 0, \qquad \sigma = \alpha, \beta, \\ (\partial_t + \mathbf{u}_{\sigma 0} \cdot \nabla)\mathbf{u}_{\sigma 0} &= -\frac{e}{m}\overline{\mathbf{E}} = \mathbf{0} \end{aligned}$$

are clearly fulfilled.

- Now consider a perturbation

$$n_{\sigma 1}(z,t) = \hat{n}_{\sigma 1}\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \qquad u_{\sigma 1}(z,t) = \hat{u}_{\sigma 1}\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \qquad \hat{n}_{\sigma 1} \ll n_{\sigma 0}, \quad \hat{u}_{\sigma 1} \ll u_{\sigma 0}.$$

- The linearized fluid equations read, again for $\sigma = \alpha, \beta$,

$$-\mathrm{i}\omega\hat{n}_{\sigma 1} + \mathrm{i}k(n_{\sigma 0}\hat{u}_{\sigma 1} + \hat{n}_{\sigma 1}u_{\sigma 0}) = 0, \tag{378}$$

$$-\mathrm{i}\omega\hat{u}_{\sigma 1} + \mathrm{i}ku_{\sigma 0}\hat{u}_{\sigma 1} = -\frac{e}{m}\hat{E}_1, \tag{379}$$

  where $\overline{\mathbf{E}}_1 = \hat{E}_1\mathbf{e}_z\ \mathrm{e}^{\mathrm{i}(kz-\omega t)}$.

- The two streams α and β are coupled through a common electric mean field amplitude $\hat{E}_1$ which is determined via Poisson's equation

$$\epsilon_0 \mathrm{i}k\hat{E}_1 = -e(\hat{n}_{\alpha 1} + \hat{n}_{\beta 1}). \tag{380}$$

- From (378) follows

$$\hat{n}_{\sigma 1} = \frac{kn_{\sigma 0}\hat{u}_{\sigma 1}}{\omega - ku_{\sigma 0}},$$

  and from (379)

$$\hat{u}_{\sigma 1} = \frac{e\hat{E}_1}{m\mathrm{i}(\omega - ku_{\sigma 0})} \qquad \Rightarrow \qquad \hat{n}_{\sigma 1} = \frac{kn_{\sigma 0}e\hat{E}_1}{m\mathrm{i}(\omega - ku_{\sigma 0})^2}$$

  so that (380) becomes

$$\hat{E}_1\left[1 - \frac{\omega_{\mathrm{p}_\alpha}^2}{(\omega - ku_{\alpha 0})^2} - \frac{\omega_{\mathrm{p}_\beta}^2}{(\omega - ku_{\beta 0})^2}\right] = 0 \tag{381}$$

  with

$$\omega_{\mathrm{p}_\sigma}^2 = \frac{e^2 n_{\sigma 0}}{\epsilon_0 m}, \qquad \sigma = \alpha, \beta.$$

- The square bracket in (381) must be zero, which gives us the dispersion relation.

- The fact that there will be an electrostatic field composed of certain modes due to the presence of the two streams physically means that kinetic energy stored in the two beams will be converted to electrostatic waves (and perhaps to electromagnetic waves if some transformation mechanism exists).

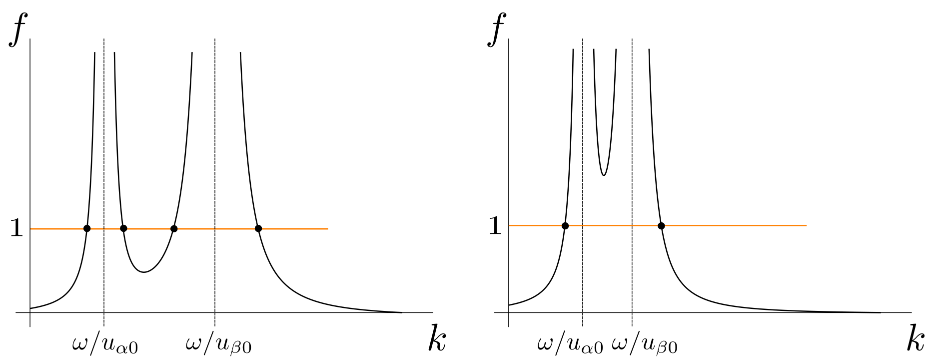
- If the generated electric field is unstable and grows it may act back on the beams, e.g., "destroying" them by making them turbulent, mix them, or bunch them.

- Solving the square bracket = 0 for ω there will in general be four roots, which may be complex, $\omega = \omega_r + \mathrm{i}\omega_i$.

- In fact, plotting

$$f(\omega, k) = \frac{\omega_{\mathrm{p}_\alpha}^2/\omega^2}{(1 - k u_{\alpha 0}/\omega)^2} + \frac{\omega_{\mathrm{p}_\beta}^2/\omega^2}{(1 - k u_{\beta 0}/\omega)^2}$$

  vs k (with the other quantities fixed) we may obtain four real roots or two real (+ two complex):



- Depending on the imaginary part of ω the respective normal mode of the electric field

  grows in time (*two-stream instability*, $\omega_i > 0$),
  is constant in time (neutral, $\omega_i = 0$),
  is damped in time (stable, $\omega_i < 0$).

- The two singularities in f occur at $k = \omega/u_{\alpha 0}$ and $k = \omega/u_{\beta 0}$. If the two beam velocities are too close together no four real roots for ω exist (right figure above).

- The unstable waves have phase velocities ω/k between $u_{\alpha 0}$ and $u_{\beta 0}$. It is not surprising that in this range the wave-particle coupling is most efficient.

- One may analyze similarly assuming ω real and k complex. Then one finds the modes growing in space (instability), neutral in space, and damped in space.

- For simplicity, consider the cases where

$$\frac{\omega_{\mathrm{p}_\alpha}^2}{u_{\alpha 0}^2} = \frac{\omega_{\mathrm{p}_\beta}^2}{u_{\beta 0}^2}.$$

This includes the important case of two homogeneous, counter-propagating streams, $n_{\alpha 0} = n_{\beta 0}$, $u_{\alpha 0} = -u_{\beta 0}$.

E Show that by introducing a harmonic average of the fluid velocities u,

$$\frac{1}{u} = \frac{1}{2}\left(\frac{1}{u_{\alpha 0}} + \frac{1}{u_{\beta 0}}\right),$$

an averaged plasma frequency ω_{p},

$$\frac{\omega_{\mathrm{p}}^2}{u^2} = \frac{\omega_{\mathrm{p}_\alpha}^2}{u_{\alpha 0}^2} = \frac{\omega_{\mathrm{p}_\beta}^2}{u_{\beta 0}^2},$$

the dimensionless frequency

$$x = \frac{\omega(u_{\beta 0} - u_{\alpha 0})}{\omega_{\mathrm{p}}(u_{\beta 0} + u_{\alpha 0})},$$

and a parameter y to parameterize the wave number,

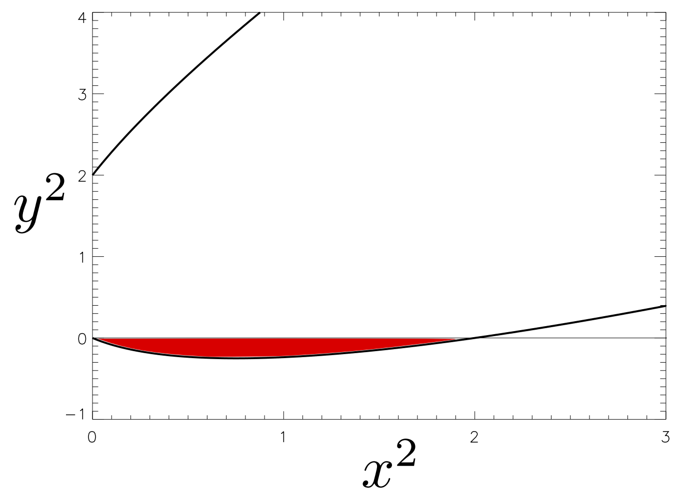
$$k = \frac{\omega}{u} + y\frac{\omega_{\mathrm{p}}}{u},$$

the dispersion relation $1 - f(\omega, k) = 0$ becomes

$$\frac{1}{(x-y)^2} + \frac{1}{(x+y)^2} = 1. \qquad (382)$$

- The fourth-order equation in y (382) has the solution

$$y^2 = x^2 + 1 \pm \sqrt{4x^2 + 1}.$$



- Consider this time ω real (i.e., x real) and k complex (i.e., y complex).
- The upper branch yields always $y^2 > 0$ and thus real k, i.e., no instabilities.
- The lower branch leads to $y^2 < 0$ for $x^2 \in]0, 2[$ (red). The two corresponding imaginary solutions of that branch with $\operatorname{Im} y \gtrless 0$ describe spatially damped and growing waves, respectively.
- We see that such damped and growing wave solutions exist down to very small frequencies $\omega \sim x$.
- The upper cut-off $x^2 = 2$ corresponds to the maximum frequency

$$\omega_{\max} = \sqrt{2}\,\omega_{\mathrm{p}}\,\frac{u_{\beta 0} + u_{\alpha 0}}{|u_{\beta 0} - u_{\alpha 0}|}$$

  for which an unstable, spatially growing wave exists.
- For small differences in the stream velocities that maximum frequency can be much greater than the plasma frequency.

E Show that the maximum *growth rate* occurs at the frequency

$$\omega_{\mathrm{max.growth}} = \frac{\sqrt{3}}{2}\omega_{\mathrm{p}}\,\frac{u_{\beta 0} + u_{\alpha 0}}{|u_{\beta 0} - u_{\alpha 0}|}.$$

- Whether an instability "survives" and becomes "dangerous" for a certain desired plasma dynamics depends on whether the growth rate is high enough to develop the instability on the time scales of interest. There might be other effects not included in the modeling (e.g., collisions, temperature, boundary effects) that may calm the instability down (or enhance it).

- We assumed cold plasma in our calculations (no pressure term) so that $u \gtrsim \sqrt{k_{\mathrm{B}}T/m}$.

- Why could we *not* care about such instabilities in our earlier fluid treatments in chapter 4?

- In fact, if, e.g., we choose species β to be much heavier than α, i.e., like a charge-neutralizing, immobile background, eq. (381) becomes

$$\hat{E}_1 \left[1 - \frac{{\omega_{\mathrm{p}}}_\alpha^2}{(\omega - k u_{\alpha 0})^2} \right] = 0$$

which leads to the earlier dispersion relation (281) (dropping α)

$$(\omega - ku)^2 = \omega_{\mathrm{p}}^2$$

without any instabilities.

E Repeat our calculations for an electron fluid moving through the heavier ion fluid with $u_{\mathrm{i},0} = 0$.

- We see in the above plot of y^2 vs x^2 that for large enough x^2 and y^2 neither frequency nor wave number are complex. Large x^2 and y^2 mean large frequencies and large wavelengths on the scales set by ω_{p}, u (and $|u_{\beta 0} - u_{\alpha 0}|$).

6.1.1 *Kinetic treatment of the two-stream instability*

- Of course, any instability that appears already in a fluid description is accesible to a kinetic analysis as well. The opposite is not true, as fluid theory deals with moments of phase space distribution functions only. We will see now that the kinetic treatment of the two-stream instability confirms the results we just obtained using fluid theory. Examples where a kinetic treatment is really mandatory to obtain an instability at all will be given below.

- Consider this time a homogeneous, field-free plasma of streaming ions and electrons at rest,

$$f_{\mathrm{i},0}(\mathbf{v}) = \delta(\mathbf{v}-\mathbf{u}), \qquad f_{\mathrm{e},0}(\mathbf{v}) = \delta(\mathbf{v}).$$

- The zeros of the dielectric function [cf. (184) for plasma without external fields] (where we take $k \geq 0$ and real and look for complex ω now),

$$D(\mathbf{k},\omega) = 1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \int \mathrm{d}v_\parallel \, \frac{\partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)}{v_\parallel - \omega/k} = 0, \quad v_\parallel = \frac{\mathbf{k}\cdot\mathbf{v}}{k}, \quad (383)$$

where

$$F^{(0)}_{\parallel\sigma}(v_\parallel) = \int \mathrm{d}^3v \, f_{\sigma 0}(\mathbf{v}) \, \delta\left(v_\parallel - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right),$$

i.e.,

$$\begin{aligned} F^{(0)}_{\parallel\mathrm{i}}(v_\parallel) &= \int \mathrm{d}^3v \, \delta(\mathbf{v}-\mathbf{u}) \, \delta\left(v_\parallel - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right) = \delta\left(v_\parallel - \frac{\mathbf{k}\cdot\mathbf{u}}{k}\right), \\ F^{(0)}_{\parallel\mathrm{e}}(v_\parallel) &= \int \mathrm{d}^3v \, \delta(\mathbf{v}) \, \delta\left(v_\parallel - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right) = \delta(v_\parallel) \end{aligned}$$

determine the modes plasma that is free from external fields.

E Show that the integrations in (383) can be performed and lead to

$$1 - \left(\frac{\omega_{\mathrm{p_e}}}{k}\right)^2 \frac{1}{(\omega/k)^2} - \left(\frac{\omega_{\mathrm{p_i}}}{k}\right)^2 \frac{1}{(\mathbf{k}\cdot\mathbf{u}/k - \omega/k)^2} = 0.$$

We thus find the analog of the square bracket in (381) (now for electrons at rest and drifting ions),

$$1 = \frac{\omega_{\mathrm{p_e}}^2}{\omega^2} + \frac{\omega_{\mathrm{p_i}}{}^2}{(\mathbf{k}\cdot\mathbf{u} - \omega)^2}. \quad (384)$$

- The plot of the right hand side (for a real k) vs ω looks qualitatively similar to the plots above for f vs k.

E Plot the right hand side of (384) vs ω (with the other quantities fixed to some reasonable values).

E Show that the local minimum of the right hand side of (384) occurs at

$$\omega_\cup = \mathbf{k}\cdot\mathbf{u} \, \frac{(\omega_{\mathrm{p_e}}/\omega_{\mathrm{p_i}})^{2/3}}{(\omega_{\mathrm{p_e}}/\omega_{\mathrm{p_i}})^{2/3} + 1}.$$

E Show that the right hand side of (384) evaluated at $\omega = \omega_{\text{U}}$ is greater than unity if

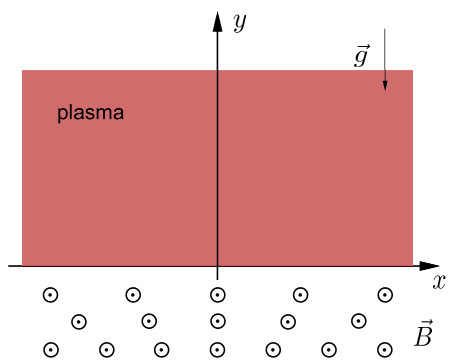
$$|\mathbf{k} \cdot \mathbf{u}| < \omega_{\text{p}_\text{e}} \left[1 + \left(\frac{\omega_{\text{p}_\text{i}}}{\omega_{\text{p}_\text{e}}} \right)^{2/3} \right]^{3/2} . \tag{385}$$

This is called the *two-stream-instability* condition.

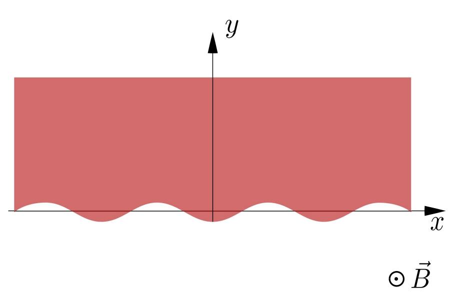
- Again, as we assumed a cold plasma the beam velocity should be much greater than the thermal velocity.

6.2 INSTABILITIES DUE TO SPATIAL CONFINEMENT

- Consider a low-density, low-temperature plasma supported against gravity by magnetic pressure $B^2/(2\mu_0)$.



- The increase in the magnetic field in $-y$-direction is such that it counterbalances the gravitational field.
- We assume that the magnetic pressure dominates the plasma pressure, $B^2/(2\mu_0) \gg p = nk_\text{B}T$. As a consequence, the magnetic field inside the plasma equals approximately the externally applied magnetic field.
- Now we consider a perturbation of the plasma at its lower boundary.

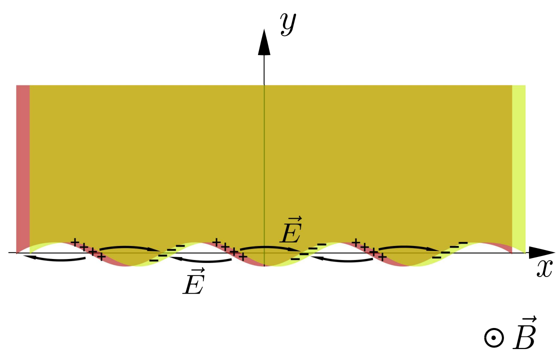


- According to (42), electrons and ions will drift (already in the unperturbed situation) according

$$\mathbf{v}_\sigma = \frac{m_\sigma \mathbf{g} \times \mathbf{B}}{q_\sigma B^2}. \tag{386}$$

- Because of the m_σ/q_σ-dependence the ions will drift (writing as usual $m_e = m$ and $m_i = M$) by M/m faster into $-x$-direction than the electrons, which drift in the $+x$-direction.

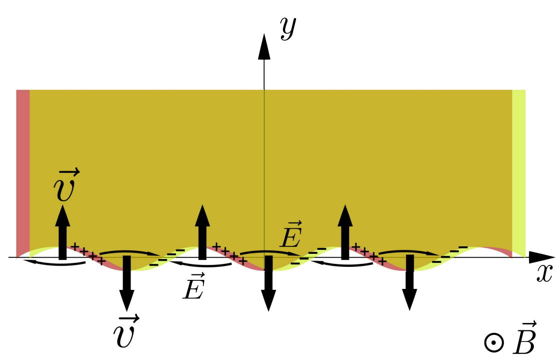
- As a result, charge separation occurs, and an electric field is created.



- This field gives rise to a charge-independent drift

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2},$$

which will act in such a way to increase the initial perturbation.



- This qualitative way of blazing the trail through a potentially unstable plasma configuration gives most physical insight but no quantitative answer for, e.g., a growth rate.

E What happens at the top surface of that plasma if we assuming the same perturbation there?

- The setup "plasma on top of a magnetic field in a gravitational field" reminds of the RAYLEIGH-TAYLOR instability for heavy (incompressible) fluids on top of lighter (incompressible) fluids in a gravitational field. The magnetic field assumes the role of the lighter fluid here.

- In inertial confinement fusion (ICF) one tries to compress plasma by a radiation field, i.e., electromagnetic waves. The same kind of instability occurs there as well, with the radiation field assuming the role of the lighter fluid (in fact, *photons* are massless).

- We consider here for simplicity the case where gravity causes the drift (386) (although gravity is usually negligible in plasma confinement experiments).

- The key element to produce the gravitational instability above was

  1. a first drift that causes charge separation,
  2. a second drift that enhances the perturbation.

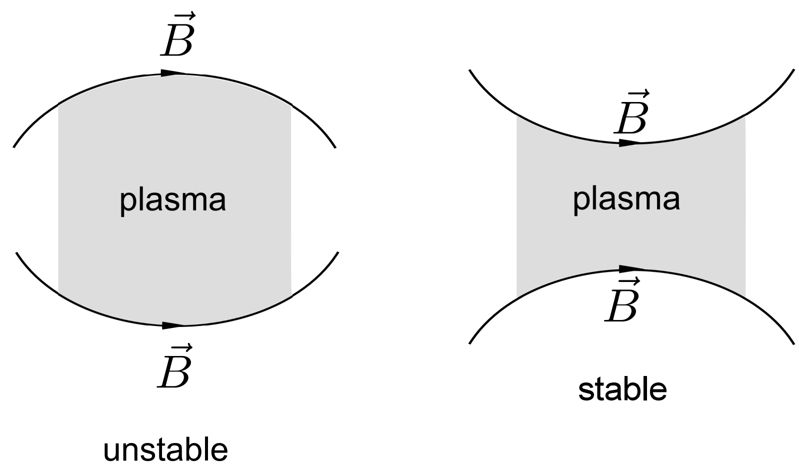
- If we replace in (42) $\mathbf{F} = m\mathbf{g}$ by the force due to the curvature of the magnetic field and its decrease in magnitude we obtain the drift [cf. (50)],

$$\mathbf{v}_{\text{gc,curv,grad}} = \frac{m}{q}\frac{\mathbf{R}\times\mathbf{B}}{R^2B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$

  which also causes a charge separation.[1]

- Hence we have the following two possible situations, depending on the curvature:

[1] Note that whenever the force does not depend on the sign of the charge the drift velocity does and vice versa.



E Why is the left configuration unstable and the right stable? Explain by sketching (in a side-view) how a perturbation develops due to the relevant drifts.

6.3 GENERAL STABILITY CONSIDERATIONS

- In the absence of external fields we can work out some general stability criteria because the dielectric function has a sufficiently simple form in this case.
- As in section 6.1.1, we are looking for zeros of the dielectric function

$$D(\mathbf{k},\omega) = 1 - \sum_\sigma \left(\frac{\omega_{p\sigma}}{k}\right)^2 \int dv_\parallel \frac{\partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)}{v_\parallel - \omega/k} = 0, \quad v_\parallel = \frac{\mathbf{k}\cdot\mathbf{v}}{k}$$

with

$$F^{(0)}_{\parallel\sigma}(v_\parallel) = \int d^3v\, f_{\sigma 0}(\mathbf{v})\, \delta\left(v_\parallel - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right),$$

specialize on electrons and singly ionized ions,

$$1 - \left(\frac{\omega_{pe}}{k}\right)^2 \int \frac{dv_\parallel}{v_\parallel - \omega/k} \frac{\partial}{\partial v_\parallel}\left[F^{(0)}_{\parallel e}(v_\parallel) + \frac{m}{M} F^{(0)}_{\parallel i}(v_\parallel)\right] = 0$$

but do not further specify the distributions

$$f_\sigma = f_{\sigma 0} + f_{\sigma 1}(\mathbf{v})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \qquad \sigma = e, i \tag{387}$$

for now.

- For

$$\omega = \omega_r + i\omega_i$$

we see from

$$\begin{aligned} 0 &= 1 - \left(\frac{\omega_{\mathrm{Pe}}}{k}\right)^2 \int \frac{dv_\parallel}{v_\parallel - (\omega_r + i\omega_i)/k} \frac{\partial}{\partial v_\parallel} \underbrace{\left[F^{(0)}_{\parallel e}(v_\parallel) + \frac{m}{M} F^{(0)}_{\parallel i}(v_\parallel)\right]}_{F^{(0)}(v_\parallel)} \\ &= 1 - \left(\frac{\omega_{\mathrm{Pe}}}{k}\right)^2 \int dv_\parallel \frac{v_\parallel - \omega_r/k - i\omega_i/k}{(v_\parallel - \omega_r/k)^2 + (\omega_i/k)^2} \frac{\partial}{\partial v_\parallel} F^{(0)}(v_\parallel) \end{aligned}$$

that real and imaginary part of this equation must vanish separately.

- The integration path must pass under the pole (cf. discussion of LANDAU contours in section 3.3.1).

- This implies that, if $\omega_i > 0$, we can perform the integral along the real $v_\parallel$-axis,

$$\begin{aligned} \int_{-\infty}^{\infty} dv_\parallel \frac{\partial_{v_\parallel} F^{(0)}(v_\parallel)}{(v_\parallel - \omega_r/k)^2 + (\omega_i/k)^2} &= 0, \\ \Rightarrow \quad 1 - \left(\frac{\omega_{\mathrm{Pe}}}{k}\right)^2 \int_{-\infty}^{\infty} dv_\parallel \frac{v_\parallel \, \partial_{v_\parallel} F^{(0)}(v_\parallel)}{(v_\parallel - \omega_r/k)^2 + (\omega_i/k)^2} &= 0. \end{aligned} \tag{388}$$

- If

$$\forall v_\parallel \; : \qquad v_\parallel \frac{\partial F^{(0)}(v_\parallel)}{\partial v_\parallel} \leq 0, \qquad v_\parallel = \frac{\mathbf{k} \cdot \mathbf{v}}{k} \tag{389}$$

equation (388) cannot be fulfilled because the integrand is negative definite.

- Hence we have proven the following:

  If (389) is fulfilled $\forall \mathbf{k}$ with

  $$F^{(0)}(v_\parallel) = \int d^3v \left[f_{e0}(\mathbf{v}) + \frac{m}{M} f_{i0}(\mathbf{v})\right] \delta\left(v_\parallel - \frac{\mathbf{k} \cdot \mathbf{v}}{k}\right)$$

  $\omega_i \leq 0$, and the plasma configuration is stable.

- The inequality (389) holds for monotonically decreasing $F^{(0)}(v_\parallel)$.

E Show that any isotropic distribution function $f_0(\mathbf{v}) = f_0(v^2)$ is stable, independent of whether it is monotonically decreasing or not.

S In this case

$$F^{(0)}(v_\parallel) = \int \mathrm{d}^2 v_\perp \, f_0(v_\parallel^2 + v_\perp^2) = 2\pi \int \mathrm{d}v_\perp \, v_\perp f_0(v_\parallel^2 + v_\perp^2)$$

and

$$\begin{aligned} v_\parallel \frac{\partial F^{(0)}(v_\parallel)}{\partial v_\parallel} &= 2v_\parallel^2 \frac{\partial F^{(0)}(v_\parallel)}{\partial v_\parallel^2} = 4\pi v_\parallel^2 \int \mathrm{d}v_\perp \, v_\perp \frac{\partial}{\partial v_\parallel^2} f_0(v_\parallel^2 + v_\perp^2) \\ &= 4\pi v_\parallel^2 \int \mathrm{d}v_\perp \, v_\perp \frac{\partial}{\partial v_\perp^2} f_0(v_\parallel^2 + v_\perp^2) = 2\pi v_\parallel^2 \int \mathrm{d}v_\perp \frac{\partial}{\partial v_\perp} f_0(v_\parallel^2 + v_\perp^2) \\ &= 2\pi v_\parallel^2 f_0 \Big|_{v_\perp = 0}^{\infty} = -2\pi v_\parallel^2 f_0(v_\parallel^2) \leq 0. \end{aligned}$$

6.4 GENTLE-BUMP INSTABILITY

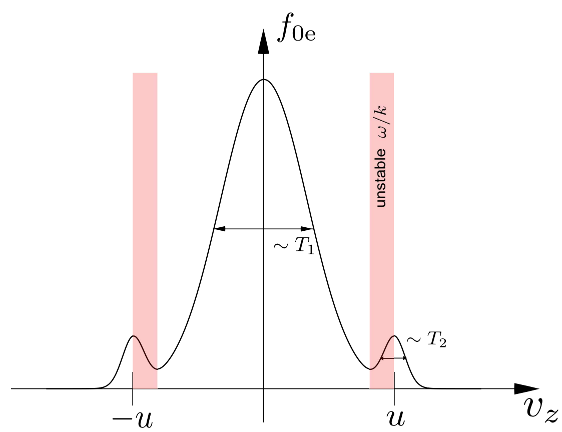
- In the cold two-stream instability there are no thermal particles that could be in resonance with the unstable waves, i.e., $v_{\mathrm{th}} \neq v_\varphi = \omega/k$.
- If such resonant effects play a role a kinetic treatment is required, and a potential instability cannot be captured by a fluid treatment.
- As an example we consider an electron distribution function with a "bump" and cold ions,

$$\begin{aligned} f_{\mathrm{e}0} &= \frac{n_1}{n_\mathrm{e}} \left(\frac{m}{2\pi k_\mathrm{B} T_1} \right)^{3/2} \exp\left(-\frac{m v^2}{2 k_\mathrm{B} T_1} \right) \\ &\quad + \frac{n_2}{n_\mathrm{e}} \delta(v_x)\delta(v_y) \left(\frac{m}{2\pi k_\mathrm{B} T_2} \right)^{1/2} \frac{1}{2} \left[\exp\left(-\frac{m(v_z - u)^2}{2 k_\mathrm{B} T_2} \right) + \exp\left(-\frac{m(v_z + u)^2}{2 k_\mathrm{B} T_2} \right) \right], \\ f_{\mathrm{i}0} &= \delta(\mathbf{v}), \end{aligned} \tag{390}$$

where

$$n_\mathrm{e} = n_1 + n_2 \gg n_2, \qquad T_2 \ll T_1, \qquad u^2 \gg \frac{2 k_\mathrm{B} T_1}{m}.$$

- There are bumps in the electron distribution function at velocities $v_z = \pm u$, describing to beams of electrons moving along the z-axis in opposite direction, besides the thermal electrons. As $n_2 \ll n_\mathrm{e}$ the bumps can be considered a perturbation. Moreover, because $T_2 \ll T_1$ the thermal spread in the beams is small compared to the spread of the main, MAXWELLian part of the distribution. Finally, $u^2 \gg 2k_\mathrm{B}T_1/m$ ensures that the bumps are well separated from the main, thermal part.



- For the distributions $F^{(0)}_{\|\mathrm{e,i}}$ we obtain

$$
\begin{aligned}
F^{(0)}_{\|\mathrm{e}} &= \int \mathrm{d}^3v\, f_{\mathrm{e}0}(\mathbf{v})\, \delta\left(v_\| - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right) \\
&= \frac{n_1}{n_\mathrm{e}}\left(\frac{m}{2\pi k_\mathrm{B}T_1}\right)^{3/2}\int \mathrm{d}^3v\,\exp\left(-\frac{mv^2}{2k_\mathrm{B}T_1}\right)\delta\left(v_\| - \frac{\mathbf{k}\cdot\mathbf{v}}{k}\right) \\
&\quad + \frac{n_2}{n_\mathrm{e}}\left(\frac{m}{2\pi k_\mathrm{B}T_2}\right)^{1/2}\frac{1}{2}\int \mathrm{d}^3v\,\delta(v_x)\delta(v_y) \\
&\quad\times\left[\exp\left(-\frac{m(v_z-u)^2}{2k_\mathrm{B}T_2}\right)+\exp\left(-\frac{m(v_z+u)^2}{2k_\mathrm{B}T_2}\right)\right]\delta\left(v_\| - \frac{k_zv_z}{k} - \frac{k_xv_x+k_yv_y}{k}\right) \\
&= \frac{n_1}{n_\mathrm{e}}\left(\frac{m}{2\pi k_\mathrm{B}T_1}\right)^{1/2}\exp\left(-\frac{mv_\|^2}{2k_\mathrm{B}T_1}\right) \\
&\quad + \frac{n_2}{n_\mathrm{e}}\left(\frac{m}{2\pi k_\mathrm{B}T_2}\right)^{1/2}\frac{1}{2}\int \mathrm{d}v_z \\
&\quad\times\left[\exp\left(-\frac{m(v_z-u)^2}{2k_\mathrm{B}T_2}\right)+\exp\left(-\frac{m(v_z+u)^2}{2k_\mathrm{B}T_2}\right)\right]\frac{k}{|k_z|}\delta\left(v_z - \frac{k}{k_z}v_\|\right) \\
&= \frac{n_1}{n_\mathrm{e}}\left(\frac{m}{2\pi k_\mathrm{B}T_1}\right)^{1/2}\exp\left(-\frac{mv_\|^2}{2k_\mathrm{B}T_1}\right) \\
&\quad + \frac{n_2}{n_\mathrm{e}}\frac{1}{2}\left(\frac{m}{2\pi k_\mathrm{B}T_2}\right)^{1/2}\frac{k}{|k_z|}\left[\exp\left(-\frac{m(kv_\|/k_z-u)^2}{2k_\mathrm{B}T_2}\right)+\exp\left(-\frac{m(kv_\|/k_z+u)^2}{2k_\mathrm{B}T_2}\right)\right], \\
F^{(0)}_{\|\mathrm{i}} &= \delta(v_\|).
\end{aligned}
$$

- The dielectric function with

$$\omega = \omega_r + \mathrm{i}\omega_i$$

then reads (cf. section 6.3)

$$\begin{aligned} D(k,\omega) &= 1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \int \mathrm{d}v_\parallel \, \frac{\partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)}{v_\parallel - \omega_r/k - \mathrm{i}\omega_i/k} \qquad (391) \\ &\simeq 1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \left\{ \mathrm{P}\int \mathrm{d}v_\parallel \, \frac{\partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)}{v_\parallel - \omega_r/k} + \mathrm{i}\pi \, \partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)\Big|_{v_\parallel = \omega_r/k} \right\}. \end{aligned}$$

In the last step we used the Plemelj formula (195),

$$\lim_{\varepsilon\to 0^+} \int \mathrm{d}u \, \frac{g(u)}{u - v_\varphi - \mathrm{i}\varepsilon} = \mathrm{P}\int \mathrm{d}u \, \frac{g(u)}{u - v_\varphi} + \pi \mathrm{i} g(u = v_\varphi),$$

assuming that $\omega_i \ll \omega_r$, which also allows to expand

$$D(k,\omega) = D(k,\omega_r) + \mathrm{i}\omega_i \, \frac{\partial D(k,\omega_r)}{\partial \omega_r}. \qquad (392)$$

- We are seeking the zeros of the dielectric function. Hence, with

$$D(k,\omega) = D_r(k,\omega) + \mathrm{i}D_i(k,\omega)$$

we have from (392),

$$0 \simeq D_r(k,\omega_r) + \mathrm{i}D_i(k,\omega_r) + \mathrm{i}\omega_i \, \frac{\partial D_r(k,\omega_r)}{\partial \omega_r}$$

and thus

$$\omega_i \simeq -\frac{D_i(k,\omega_r)}{\frac{\partial D_r(k,\omega_r)}{\partial \omega_r}}, \qquad (393)$$

$$D_r(k,\omega) = 1 - \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \mathrm{P}\int \mathrm{d}v_\parallel \, \frac{\partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)}{v_\parallel - \omega_r/k} = 0. \qquad (394)$$

Moreover, from (391),

$$D_i(k,\omega) = -\pi \sum_\sigma \left(\frac{\omega_{\mathrm{p}\sigma}}{k}\right)^2 \partial_{v_\parallel} F^{(0)}_{\parallel\sigma}(v_\parallel)\Big|_{v_\parallel = \omega_r/k}$$

$$= \quad -\pi \left(\frac{\omega_{\mathrm{Pe}}}{k}\right)^2 \partial_{v_\parallel} \underbrace{\left(F^{(0)}_{\parallel \mathrm{e}}(v_\parallel) + \frac{m}{M} F^{(0)}_{\parallel \mathrm{i}}(v_\parallel)\right)}_{F^{(0)}_\parallel}\Bigg|_{v_\parallel=\omega_r/k}$$

$$= \quad -\pi \left(\frac{\omega_{\mathrm{Pe}}}{k}\right)^2 \partial_{v_\parallel} F^{(0)}_\parallel (v_\parallel)\Big|_{v_\parallel=\omega_r/k} \tag{395}$$

so that for (393)

$$\boxed{\omega_i \simeq \pi \left(\frac{\omega_{\mathrm{Pe}}}{k}\right)^2 \frac{\partial_{v_\parallel} F^{(0)}_\parallel (v_\parallel)\Big|_{v_\parallel=\omega_r/k}}{\partial D_r(k,\omega_r)/\partial\omega_r}} \,. \tag{396}$$

- We need to calculate $D_r(k,\omega_r)$ and subsequently $\partial D_r(k,\omega_r)/\partial\omega_r$. Now, for $v_\varphi = \omega/k \gg v_\parallel$ with most contributions to the integral for $v_\parallel \simeq v_{\mathrm{th}} = \sqrt{2k_{\mathrm{B}}T_1/m}$ we can expand (394), as in section 3.3.2

$$\mathrm{P}\int \mathrm{d}v_\parallel \frac{\partial_{v_\parallel} F^{(0)}_\parallel (v_\parallel)}{\omega_r/k - v_\parallel} \simeq \int \mathrm{d}v_\parallel\, \partial_{v_\parallel} F^{(0)}_\parallel (v_\parallel) \left(\frac{k}{\omega_r} + \frac{k^2 v_\parallel}{\omega_r^2} + \frac{k^3 v_\parallel^2}{\omega_r^3} + \cdots\right).$$

E Calculate $D_r(k,\omega_r)$.

E Calculate ω_r^2 up to the first correction beyond $\omega_r^2 \simeq \omega_{\mathrm{Pe}}^2$.

S As in section 3.3.2, we find the BOHM-GROSS result

$$\omega_r^2 \simeq \omega_{\mathrm{Pe}}^2 (1 + 3k^2\lambda_{\mathrm{D1}}^2)$$

where λ_{D1} is the DEBYE length evaluated with $k_{\mathrm{B}}T_1$.

- With $\omega_{\mathrm{Pi}}^2 \ll \omega_{\mathrm{Pe}}^2$ we have in leading order just the usual

$$D_r(k,\omega_r) \simeq 1 - \frac{\omega_{\mathrm{Pe}}^2}{\omega_r^2}$$

so that $\partial D_r(k,\omega_r)/\partial\omega_r > 0$.

- The sign of ω_i is thus, according (396), determined by *the slope of* $F^{(0)}_\parallel (v_\parallel)$ *at* $v_\parallel = \omega_r/k$, i.e., where the electron velocity equals the phase velocity of the wave.

- Such particles for which $v_\parallel = v_\varphi$ are called *resonant particles*. These particles may efficiently couple to the wave, subtracting or feeding energy into it, which leads to (Landau-) damping ($\omega_i < 0$) or an instability ($\omega_i > 0$), respectively.

E Using $n_1 \gg n_2$, $n_1 k_\mathrm{B} T_1 \gg n_2 m u^2$, $\omega_{\mathrm{p}1}^2 = e^2 n_1/(\epsilon_0 m)$, show that

$$\begin{aligned}\omega_i \quad &\simeq \quad -\sqrt{\frac{\pi}{8}}\frac{\omega_{\mathrm{p}1}}{(k\lambda_{\mathrm{D}1})^3}\exp\left(-\frac{1}{2(k\lambda_{\mathrm{D}1})^2}-\frac{3}{2}\right) \\ &\quad +\frac{n_2}{n_1}\left(\frac{T_1}{T_2}\right)^{3/2}\frac{k^3}{k_z^3}\left(\frac{k_z u}{\omega_r}-1\right)\exp\left[-\frac{T_1/T_2}{2(k_z\lambda_{\mathrm{D}1})^2}\left(1-\frac{k_z u}{\omega_r}\right)^2\right].\end{aligned} \tag{397}$$

- We see:

1. The first term < 0 is the Landau damping contribution we know already from section 3.3.2.
2. The second term originates from the non-Maxwellian bump-part. Obviously, the ratio of u and ω_r/k_z matters. If

$$\frac{\omega_r}{k_z} < u \quad \Rightarrow \quad 1 < u\frac{k_z}{\omega_r} \quad \Rightarrow \quad u\frac{k_z}{\omega_r} - 1 > 0$$

   the second term increases ω_i. Particles give energy to the wave. If the second term is greater than the first there is an instability.
3. If $\frac{\omega_r}{k_z} > u$ the second term enhances damping. The wave looses energy to the particles.
4. This maximum rate is found for

$$\left|\frac{k_z u}{\omega_{\mathrm{p}1}}\right| = 1 + \sqrt{(k_z\lambda_{\mathrm{D}1})^2\frac{T_2}{T_1}} \tag{398}$$

   to be

$$\omega_{i,\max} = \sqrt{\frac{\pi}{8}}\frac{\omega_{\mathrm{p}1}}{(k\lambda_{\mathrm{D}1})^3}\left[\frac{n_2}{n_1}(k\lambda_{\mathrm{D}1})^3\frac{mu^2}{k_\mathrm{B}T_2}\,\mathrm{e}^{-1/2} - \exp\left(-\frac{1}{2(k\lambda_{\mathrm{D}1})^2}-\frac{3}{2}\right)\right]. \tag{399}$$

5. Hence, $\omega_{i,\max} > 0$ is the *gentle-bump instability* condition.
6. The maximum growth rate increases by (i) increasing n_2 (more particles in the beams), (ii) decreasing T_2 (making the beams colder), (iii) making the beams faster.

7. The damping term in (399) is smallest for small k, i.e., long wavelengths. However, (398) restricts $|k_z|$ to values $|k_z| \approx \omega_{p_1}/|u|$. As $k \geq |k_z|$ the most unstable waves propagate $\parallel u$.
8. Comparing our result (399) with the growth rate for the corresponding cold two-beam result one can show that the gentle-bump instability grows slower. However, the gentle-bump instability may turn into a two-stream instability as the bump grows and narrows. A nonlinear treatment is necessary to study such dynamics, as $n_2 \ll n_1$ would be violated at some point.

- We assumed in the above expansion of $D_r(k, \omega)$ that $\omega_r/k \gg \sqrt{2k_B T_1/m}$, which implies that the bump sits on top of the tail of the main, Maxwellian part of the distribution function. In fact, one can show that a distribution function with a small bump in a region where $\omega_r/k \lesssim \sqrt{2k_B T_1/m}$ is stable.

Why is the two-stream instability accessible to a fluid treatment?

- ... and the gentle-bump instability not?
- The two-stream instability does not require resonant particles. In fact, in the limit of δ-like bumps there are no resonant particles (red-shaded area in the figure above) at all.
- The two-stream instability is a non-resonant phenomenon whose growth is not dependent on the number of particles with $v \approx v_\varphi$.
- Assume a spontaneous increase of electron density somewhere in the plasma.[2] The beam passing over that clumped region will be slowed down, which leads (due to continuity) to an even increased density, and so on.

6.5 WEIBEL INSTABILITY

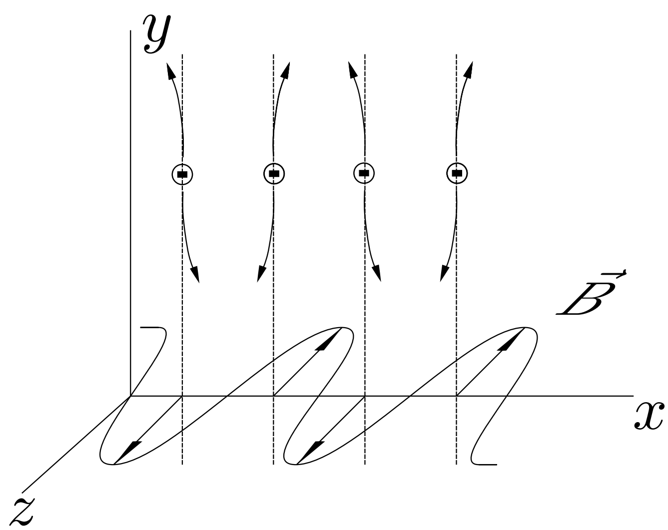
- Consider a non-isotropic electron plasma that is cold in x and z direction but hotter in its y component.
- We assume a magnetic field

$$\mathbf{B} = B_z \mathbf{e}_z \sin kx$$

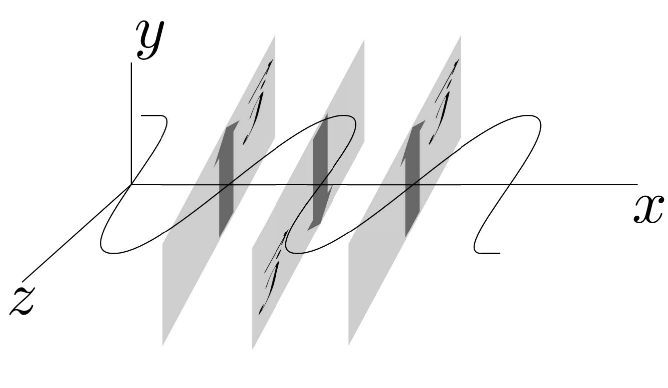
[2] One may synthesize such a perturbation by the proper combination of electrostatic waves.

builds up spontaneously.

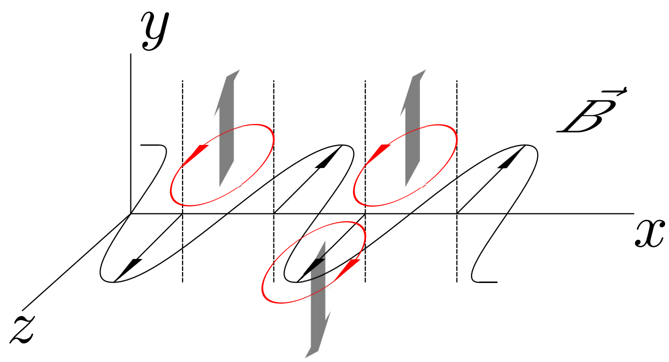
- The Lorentz force will bend the thermally moving electrons as depicted in the following figure.



- This leads to a charge bunching, resulting in current sheets:



- It is easy to see that the magnetic field generated by these current sheets (drawn red in the following figure) "feeds" the originally assumed field. Hence the magnetic field will grow.



- This instability is called the Weibel instability, the intuitive physical interpretation given here is due to Fried.

- For the simple "cold case" the growth rate is $\omega_i \simeq \omega_{\mathrm{p}} v_{\mathrm{th}}/c$.

This chapter could only bring a very limited introduction into plasma instabilities. There are many more of them (e.g., Buneman, flux tube (kink/sausage), Kelvin-Helmholtz, firehose, mirror, ...) with which one could fill an entire semester. There is also much more interesting theory on the subject (Nyquist method, Penrose criterion, ...).

7 FLUCTUATIONS

- Averaged quantities such as density, pressure, temperature etc. in local thermal equilibrium fluctuate around their respective mean values.
- One might think that these *fluctuations* are not of interest because of their smallness in the thermodynamic limit (cf. central limit theorem).
- However, plasma instabilities arise because certain *modes* of these fluctuations may grow *exponentially* in time (see previous chapter).
- Moreover, we will see in this chapter that these fluctuations contain a wealth of information.

- Consider N point-like particles in a cubic volume $V = L^3$. The fine-grained density is

$$n(\mathbf{r}, t) = \sum_{i=1}^{N} \delta[\mathbf{r} - \mathbf{r}_i(t)], \tag{400}$$

and the fluctuation around the mean value $n = N/V$ reads

$$\delta n(\mathbf{r}, t) = n(\mathbf{r}, t) - n. \tag{401}$$

Density fluctuations $\delta n(\mathbf{r}, t) \neq 0$ arise because of spatial or temporal deviations from the mean density n.

- Assuming periodic boundary conditions ($k_x L = j_x 2\pi$, $j_x \in \mathbb{Z}$ and analogous for y and z) we Fourier-transform spatially

$$\begin{aligned} \delta n_{\mathbf{k}}(t) &= \int_V \mathrm{d}^3 r \, \delta n(\mathbf{r}, t) \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\ &= \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i(t)} - N\delta(\mathbf{k}, \mathbf{0}) \end{aligned}$$

with $\delta(\mathbf{k}, \mathbf{0}) = \delta_{k_x 0}\delta_{k_y 0}\delta_{k_z 0}$. In the last step we used $\int_V \mathrm{d}^3 r \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = V\delta(\mathbf{k}, \mathbf{0})$.

- Obviously, $[\delta n_{\mathbf{k}}(\mathbf{r}, t)]^* = \delta n_{-\mathbf{k}}(\mathbf{r}, t)$.

- The *static structure factor* (aka *static form factor*) is defined as

$$\boxed{S(\mathbf{k}) = \frac{1}{N}\left\langle |\delta n_{\mathbf{k}}(t)|^2 \right\rangle = \frac{1}{N}\left\langle \delta n_{\mathbf{k}}(t)\, \delta n_{-\mathbf{k}}(t) \right\rangle}\,. \tag{402}$$

Here, $\langle\cdot\rangle$ denotes the statistical average with respect to some distribution function describing the ensemble (cf. chapter 3). We assume that the system is "stationary" in the sense that the result for $S(\mathbf{k})$ does not depend on the time at which we evaluate the right hand side of (402). Hence, we may as well write

$$S(\mathbf{k}) = \frac{1}{N}\left\langle \delta n_{\mathbf{k}}(0)\, \delta n_{-\mathbf{k}}(0) \right\rangle. \tag{403}$$

- Moreover, we can rewrite

$$\begin{aligned} S(\mathbf{k}) &= \frac{1}{N}\left\langle \left[\sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i} - N\delta(\mathbf{k},\mathbf{0})\right]\left[\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_j} - N\delta(\mathbf{k},\mathbf{0})\right]\right\rangle \\ &= \frac{1}{N}\left\langle -\sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i} N\delta(\mathbf{k},\mathbf{0}) - N\delta(\mathbf{k},\mathbf{0})\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_j} \right. \\ &\qquad \left. + \sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_i)} + N^2\delta(\mathbf{k},\mathbf{0})\delta(\mathbf{k},\mathbf{0}) \right\rangle \\ &= -N\delta(\mathbf{k},\mathbf{0}) - N\delta(\mathbf{k},\mathbf{0}) + \frac{1}{N}\left\langle \sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_i)} \right\rangle + N\delta(\mathbf{k},\mathbf{0}) \\ &= -N\delta(\mathbf{k},\mathbf{0}) + 1 + \frac{1}{N}\left\langle \sum_{i=1}^{N}\sum_{j\neq i}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_i)} \right\rangle. \end{aligned} \tag{404}$$

Here, $\mathbf{r}_{i,j} = \mathbf{r}_{i,j}(0)$ (or any other arbitrary time).

- Obviously,

$$S(\mathbf{k}) = S(-\mathbf{k}).$$

- For uncorrelated particles the last term in (404) vanishes if $\mathbf{k} \neq \mathbf{0}$. If $\mathbf{k} = \mathbf{0}$ it yields $\frac{1}{N}N(N-1) = N-1$. Hence $S(\mathbf{k}) = -N\delta(\mathbf{k},\mathbf{0}) + 1 + (N-1)\delta(\mathbf{k},\mathbf{0})$ and we obtain

$$S(\mathbf{k}) = 1 - \delta(\mathbf{k},\mathbf{0}) \tag{405}$$

for uncorrelated particles.

- In (116) we defined the single-particle distribution function as

$$f_1(X,t) = \frac{1}{n}\langle K(X,t)\rangle$$

where $K(X,t) = \sum_{i=1}^{N} \delta[X - X_i(t)]$ is the KLIMONTOVICH distribution function.

- Analogously to (401) we introduce the fluctuation of the KLIMONTOVICH distribution function

$$\delta K(X,t) = K(X,t) - \big\langle K(X,t)\big\rangle = K(X,t) - nf_1(X)$$

so that for a homogeneous system, where $f_1(X) = f_1(\mathbf{v})$,

$$\delta n(\mathbf{r},t) = \int \mathrm{d}^3 v\, \delta K(X,t)$$

and

$$\delta n_{\mathbf{k}}(t) = \int_V \mathrm{d}^3 r \int \mathrm{d}^3 v\, \delta K(X,t)\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \int \mathrm{d}^6 X\, \delta K(X,t)\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

- The static structure factor can then be written as

$$\begin{aligned} S(\mathbf{k}) &= \frac{1}{N}\Big\langle \int_V \mathrm{d}^3 r \int \mathrm{d}^3 v \int_V \mathrm{d}^3 r' \int \mathrm{d}^3 v'\, \delta K(X,t)\delta K(X',t)\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\Big\rangle \\ &= \frac{1}{N}\int \mathrm{d}^6 X' \int \mathrm{d}^6 X\, \big\langle \delta K(X,t)\delta K(X',t)\big\rangle\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \qquad (406) \\ &= \frac{1}{N}\int \mathrm{d}^3 v' \int \mathrm{d}^3 v \int_V \mathrm{d}^3 (r-r') \int_V \mathrm{d}^3 (r+r')/2\, \big\langle \delta K(X,t)\delta K(X',t)\big\rangle\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \end{aligned}$$

For translationally invariant systems the integrand can only depend on $\mathbf{r}-\mathbf{r}'$ and must not depend on $(\mathbf{r}+\mathbf{r}')/2$ so that

$$S(\mathbf{k}) = \frac{1}{n}\int \mathrm{d}^3 v' \int \mathrm{d}^3 v \int_V \mathrm{d}^3 (r-r')\, \big\langle \delta K(X,t)\delta K(X',t)\big\rangle\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}.$$

- We now want to express the average $\big\langle \delta K(X,t)\delta K(X',t)\big\rangle$ in terms of the single particle distribution function and the pair correlation function,

$$F(X) = f_1(X), \qquad G(X,X') = f_2(X,X') - F(X)F(X'),$$

where $f_2(X,X')$ is defined via (see chapter 3)

$$\langle K(X,t)K(X',t)\rangle = \delta(X-X')nf_1(X) + n^2 f_2(X,X').$$

We find

$$\begin{aligned}\langle \delta K(X,t)\delta K(X',t)\rangle &= \langle [K(X,t)-nF(X)][K(X',t)-nF(X')]\rangle \\ &= \langle K(X,t)K(X',t)\rangle - \langle K(X,t)\rangle nF(X') - nF(X)\langle K(X',t)\rangle + n^2F(X)F(X') \\ &= \delta(X-X')nF(X) + n^2 f_2(X,X') - n^2F(X)F(X') \\ &= \delta(X-X')nF(X) + n^2[G(X,X') + F(X)F(X')] - n^2F(X)F(X') \\ &= \delta(X-X')nF(X) + n^2G(X,X').\end{aligned}$$

- We thus have for (406)

$$\begin{aligned}S(\mathbf{k}) &= \frac{1}{N}\int \mathrm{d}^6X' \int \mathrm{d}^6X\ [\delta(X-X')nF(X) + n^2G(X,X')]\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \\ &= 1 + \frac{n^2}{N}\int \mathrm{d}^6X' \int \mathrm{d}^6X\ G(X,X')\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \\ &= 1 + n\int \mathrm{d}^3v' \int \mathrm{d}^3v \int_V \mathrm{d}^3(r-r')\ G(X,X')\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')},\end{aligned}$$

i.e.,

$$S(\mathbf{k}) = 1 + n\int \mathrm{d}^3v' \int \mathrm{d}^3v\, G_{\mathbf{k}}(\mathbf{v},\mathbf{v}') \tag{407}$$

with $G_{\mathbf{k}}(\mathbf{v},\mathbf{v}')$ the Fourier transform of $G(X,X')$ (with respect to $\mathbf{r}-\mathbf{r}'$), and the same argument as above concerning the $\int \mathrm{d}^3(r+r')/2$-integration.

- Defining

$$p(\mathbf{r}-\mathbf{r}') = \int \mathrm{d}^3v' \int \mathrm{d}^3v\, G(X,X')$$

we can also write (407) in the form

$$S(\mathbf{k}) = 1 + n\int_V \mathrm{d}^3r\ p(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}. \tag{408}$$

E Show that $p(\mathbf{r}-\mathbf{r}') + 1$ is the conditional probability to find a particle within a volume V/N at $\mathbf{r}$, provided there is one at $\mathbf{r}'$.

- Expectation values for observables may be expressed in terms of the structure factor. Consider, e.g., the many-body Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i} v(\mathbf{r}_j - \mathbf{r}_i)$$

for some interaction $v(\mathbf{r}_j - \mathbf{r}_i)$.

- Using the FOURIER expansion

$$v(\mathbf{r}_j - \mathbf{r}_i) = \frac{1}{V}\sum_{\mathbf{k}} v_{\mathbf{k}}\ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}, \qquad v_{\mathbf{k}} = \int_V \mathrm{d}^3 r\, v(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$$

we can write

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2V}\sum_{i=1}^{N}\sum_{j\neq i}\sum_{\mathbf{k}}{}' v_{\mathbf{k}}\ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}$$

where $\sum_{\mathbf{k}}{}'$ means that the $\mathbf{k} = \mathbf{0}$-term has been excluded from the sum.

E What does it mean physically to exclude the $\mathbf{k} = \mathbf{0}$-term from the sum?

Correlation energy density

- For the energy density we obtain, using (404), (408), and our knowledge from Statistical Physics that $\left\langle \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \right\rangle = \frac{3}{2} N k_\mathrm{B} T$,

$$\begin{aligned}
\frac{\langle H\rangle}{V} &= \frac{1}{V}\left\langle \sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}\right\rangle + \frac{1}{2V^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\sum_{\mathbf{k}}{}' v_{\mathbf{k}}\left\langle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\right\rangle \\
&= \frac{3}{2} n k_\mathrm{B} T + \frac{N}{2V^2}\sum_{\mathbf{k}}{}' v_{\mathbf{k}}[S(\mathbf{k}) + N\delta(\mathbf{k},0) - 1] \\
&= \frac{3}{2} n k_\mathrm{B} T + \frac{n}{2V}\sum_{\mathbf{k}}{}' v_{\mathbf{k}}[S(\mathbf{k}) - 1] \\
&= \frac{3}{2} n k_\mathrm{B} T + \frac{n^2}{2V}\sum_{\mathbf{k}}{}' v_{\mathbf{k}}\int_V \mathrm{d}^3 r\ p(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\
&= \frac{3}{2} n k_\mathrm{B} T + \frac{n^2}{2V}\sum_{\mathbf{k}}{}' \int_V \mathrm{d}^3 r'\, v(\mathbf{r}')\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\int_V \mathrm{d}^3 r\ p(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\
&= \frac{3}{2} n k_\mathrm{B} T + \frac{n^2}{2}\int_V \mathrm{d}^3 r\, v(\mathbf{r}) p(\mathbf{r}).
\end{aligned} \tag{409}$$

In the last step we used $\sum_{\mathbf{k}}{}' \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = V\,\delta(\mathbf{r})$ and $p(\mathbf{r}) = p(-\mathbf{r})$.

- The last term in the energy density (409) is due to correlation and may be called the *correlation energy density*.

Connection with the scattering cross section—an example

- In the Advanced Quantum Theory lecture we have learned that the amplitude for elastic scattering off a potential $v(\mathbf{r})$ in first Born approximation is
$$f_k(\Omega) = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3r\, v(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}, \qquad \mathbf{q} = \mathbf{k}_\mathrm{f} - \mathbf{k}_\mathrm{i}, \qquad k = |\mathbf{k}_\mathrm{i}| = |\mathbf{k}_\mathrm{f}|,$$
where $v(\mathbf{r})$ is the scattering potential, and $\hbar\mathbf{q}$ is the momentum transfer.
Consider some potential, e.g., that of a lattice with N ions at positions $\mathbf{R}_i$,
$$v(\mathbf{r}) = \sum_i u(\mathbf{r} - \mathbf{R}_i)$$
with $u(\mathbf{r})$ the potential of an individual ion. In analogy to the above $v_\mathbf{k}$ we write
$$u(\mathbf{r}) = \frac{1}{V}\sum_\mathbf{k} u_\mathbf{k}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \qquad u_\mathbf{k} = \int_V \mathrm{d}^3r\, u(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$
Hence
$$\begin{aligned}
f_k(\Omega) &= -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3r \sum_{i=1}^N u(\mathbf{r}-\mathbf{R}_i)\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \\
&= -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3r \sum_{i=1}^N \frac{1}{V}\sum_\mathbf{k} u_\mathbf{k}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_i)}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \\
&= -\frac{m}{2\pi\hbar^2}\sum_{i=1}^N\sum_\mathbf{k} u_\mathbf{k}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{R}_i}\delta(\mathbf{k},\mathbf{q}) \\
&= -\frac{m}{2\pi\hbar^2}u_\mathbf{q}\sum_{i=1}^N \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{R}_i} \\
\Rightarrow \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_k = |f_k(\Omega)|^2 &= \frac{m^2}{(2\pi)^2\hbar^4}|u_\mathbf{q}|^2\sum_{ji}\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot(\mathbf{R}_i-\mathbf{R}_j)} \\
&= \frac{m^2}{(2\pi)^2\hbar^4}|u_\mathbf{q}|^2 N\Bigg\{\underbrace{1}_{\text{incoherent part}} + \underbrace{\frac{1}{N}\sum_{j\neq i}\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot(\mathbf{R}_i-\mathbf{R}_j)}}_{\text{coherent part (interferences)}}\Bigg\}.
\end{aligned}$$
So far the ion positions were frozen. If they fluctuate because of thermal motion one needs to take the ensemble average, leading to
$$\left\langle \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_k \right\rangle = \frac{m^2}{(2\pi)^2\hbar^4}|u_\mathbf{q}|^2 N\left\{S(\mathbf{q}) + N\delta(\mathbf{q},\mathbf{0})\right\}.$$

This shows the relation between the structure factor and the scattering cross section. In other words, the structure factor is directly accessible through appropriate scattering experiments.

7.1 DYNAMIC STRUCTURE FACTOR

- The *dynamic structure factor* (aka *dynamic form factor*) is defined as

$$\boxed{S(\mathbf{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \left\langle \delta n_{\mathbf{k}}(t)\, \delta n_{-\mathbf{k}}(0) \right\rangle \mathrm{e}^{\mathrm{i}\omega t}}\,. \tag{410}$$

  Because we assume that the system is stationary it does not matter where the time origin is chosen. We could as well write $\langle \delta n_{\mathbf{k}}(t+\tau)\, \delta n_{-\mathbf{k}}(\tau)\rangle$ for any time τ in the integrand.

- The dynamic structure factor is the Fourier transform of the *autocorrelation function* $\langle \delta n_{\mathbf{k}}(t)\, \delta n_{-\mathbf{k}}(0)\rangle$. It may also be viewed as the *power spectrum of the density fluctuations*, given that we started from (400) and (401). The fact that both autocorrelation function and power spectrum are related is the essence of the so-called Wiener-Khinchine theorem.

- We expect that the dynamic structure factor contains even more information than the static structure factor because of the temporal information included by the different time arguments 0 and t in (410).

E Show that

$$S(\mathbf{k},\omega)\,(2\pi)^2 \delta(\omega-\omega') = \langle \delta n(\mathbf{k},\omega)\, \delta n(-\mathbf{k},-\omega')\rangle, \tag{411}$$

where

$$\delta n(\mathbf{k},\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \delta n_{\mathbf{k}}(t)\; \mathrm{e}^{\mathrm{i}\omega t}.$$

E Show that the dynamic structure factor is real and non-negative.

E Show that the dynamic structure factor fulfills the *sum rule*

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, S(\mathbf{k},\omega) = NS(\mathbf{k}). \tag{412}$$

S Integrating (411) over ω' yields

$$S(\mathbf{k},\omega)\,(2\pi)^2 \;=\; \left\langle \delta n(\mathbf{k},\omega) \int \mathrm{d}\omega'\, \delta n(-\mathbf{k},-\omega')\right\rangle$$

$$\begin{aligned} &= \left\langle \delta n(\mathbf{k},\omega) \int \mathrm{d}\omega' \int_{-\infty}^{\infty} \mathrm{d}t\, \delta n_{-\mathbf{k}}(t)\ \mathrm{e}^{-\mathrm{i}\omega' t} \right\rangle \\ &= 2\pi \left\langle \delta n(\mathbf{k},\omega)\, \delta n_{-\mathbf{k}}(0) \right\rangle \\ &= 2\pi \left\langle \int_{-\infty}^{\infty} \mathrm{d}t\, \delta n_{\mathbf{k}}(t)\ \mathrm{e}^{\mathrm{i}\omega t}\, \delta n_{-\mathbf{k}}(0) \right\rangle \end{aligned}$$

Integrating also over ω we find

$$(2\pi)^2 \int_{-\infty}^{\infty} \mathrm{d}\omega\, S(\mathbf{k},\omega) \quad = \quad (2\pi)^2 \left\langle \delta n_{\mathbf{k}}(0) \delta n_{-\mathbf{k}}(0) \right\rangle,$$

and with (403) the claimed (412).

E Analogously to (406), show that

$$\begin{aligned} S(\mathbf{k},\omega) &= \frac{V}{2\pi} \int \mathrm{d}^3 v' \int \mathrm{d}^3 v \int_V \mathrm{d}^3(r-r') \int \mathrm{d}t \left\langle \delta K(X,t) \delta K(X',0) \right\rangle \mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega t]} \\ &= \frac{1}{2\pi} \int \mathrm{d}^6 X' \int \mathrm{d}^6 X \int \mathrm{d}t \left\langle \delta K(X,t) \delta K(X',0) \right\rangle \mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega t]}. \end{aligned} \tag{413}$$

- Analogously to (408) we want $S(\mathbf{k},\omega)$ to be of the form

$$\boxed{S(\mathbf{k},\omega) = \frac{N}{2\pi} \int_V \mathrm{d}^3 r \int \mathrm{d}t \left[p_s(\mathbf{r},t) + n p(\mathbf{r},t) \right]\ \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}}\,. \tag{414}$$

  This is the case if

$$\frac{1}{n^2} \int \mathrm{d}^3 v' \int \mathrm{d}^3 v \left\langle \delta K(X,t) \delta K(X',0) \right\rangle = \frac{1}{n}\, p_s(\mathbf{r},t) + p(\mathbf{r},t). \tag{415}$$

- Integrating (414) over ω yields with (408)

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}\omega\, S(\mathbf{k},\omega) = N \int_V \mathrm{d}^3 r \left[p_s(\mathbf{r},0) + n p(\mathbf{r},0) \right]\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \quad &\overset{!}{=} \quad N S(\mathbf{k}) \\ &= \quad N + nN \int_V \mathrm{d}^3 r\ p(\mathbf{r})\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \end{aligned}$$

  and we see that, indeed, $p(\mathbf{r},0) = p(\mathbf{r})$ and

$$\int_V \mathrm{d}^3 r\, p_s(\mathbf{r},0)\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = p_{s\mathbf{k}}(0) = 1.$$

E Show that $p_s(\mathbf{r},t)\,\mathrm{d}^3 r$ is the conditional probability to find a particle at time t in $[\mathbf{r}, \mathbf{r} + (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)]$ if it was at $t = 0$ at the origin (the subscript s stands for "self").

E Show that $p(\mathbf{r},t)+1$ is the conditional probability to find a particle at time t in a volume $V/N = 1/n$ around $\mathbf{r}$ if a particle was at time $t=0$ at the origin.

- For uncorrelated particles only the term with p_s remains, the term with p vanishes.

- The question arises how to actually calculate the dynamic structure factor.

- In situations of thermodynamic equilibrium we can use the *fluctuation-dissipation theorem* for establishing a very general connection between the dynamic structure factor, the linear response, and the dielectric function.

- As we have learned how to calculate the dielectric function (at least in principle and for certain simple cases) we can then calculate the dynamic structure factor as well.

7.2 FLUCTUATION-DISSIPATION THEOREM

- We define the variation of a physical quantity B in general as

$$\delta B(\mathbf{r},t) = \langle B(\mathbf{r},t;a)\rangle - \langle B(\mathbf{r},t;0)\rangle \tag{416}$$

where $a(\mathbf{r},t)$ is a perturbation that is zero for $t < t_0$.

- We know that in first-order perturbation theory a quantum mechanical wavefunction evolves in time (see, e.g., the Advanced Quantum Theory lecture notes) in the interaction picture according

$$|\psi(t)\rangle = |\psi_0\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\, H'(t')\,|\psi_0\rangle \tag{417}$$

where we write the perturbing part $H'(t)$ in the HAMILTONian $H(t) = H_0 + H'(t)$ in the form

$$H'(t) = -\int \mathrm{d}^3 r\, A(\mathbf{r},t)a(\mathbf{r},t). \tag{418}$$

E Give examples for such couplings of a perturbation a to a quantum mechanical system via an operator A.

- The variation $\delta B(\mathbf{r},t)$ according (416) with $\langle\cdot\rangle$ interpreted as the (pure-state) quantum mechanical expectation value reads up to first order, with (417),[1]

$$\begin{aligned}
\delta B(\mathbf{r},t) &= \langle\psi(t)|B(\mathbf{r},t)|\psi(t)\rangle - \langle\psi_0|B(\mathbf{r},t)|\psi_0\rangle \\
&= \left\{\langle\psi_0| + \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\langle\psi_0|H'(t')\right\} B(\mathbf{r},t)\left\{|\psi_0\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,H'(t')\,|\psi_0\rangle\right\} \\
&\qquad -\langle\psi_0|B(\mathbf{r},t)|\psi_0\rangle \\
&= -\frac{\mathrm{i}}{\hbar}\langle\psi_0|B(\mathbf{r},t)\int_{t_0}^{t}\mathrm{d}t'\,H'(t')\,|\psi_0\rangle + \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\langle\psi_0|H'(t')B(\mathbf{r},t)\,|\psi_0\rangle \\
&= -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\langle\psi_0|[B(\mathbf{r},t),H'(t')]\,|\psi_0\rangle \\
&= \frac{\mathrm{i}}{\hbar}\int\mathrm{d}^3r'\int_{t_0}^{t}\mathrm{d}t'\,\langle\psi_0|[B(\mathbf{r},t),A(\mathbf{r}',t')]\,|\psi_0\rangle\,a(\mathbf{r}',t') \\
&= \frac{\mathrm{i}}{\hbar}\int\mathrm{d}^3r'\int_{t_0}^{t}\mathrm{d}t'\,\langle\psi_0|[B(\mathbf{r},t-t'),A(\mathbf{r}')]\,|\psi_0\rangle\,a(\mathbf{r}',t').
\end{aligned}$$

[E] Justify the last step.

7.2.1 *Linear response function*

- Introducing the *linear response function*

$$\chi_{BA}(\mathbf{r},\mathbf{r}';t-t') = \frac{\mathrm{i}}{\hbar}\Theta(t-t')\langle\psi_0|[B(\mathbf{r},t-t'),A(\mathbf{r}')]\,|\psi_0\rangle \tag{419}$$

we can write

$$\delta B(\mathbf{r},t) = \int_{-\infty}^{\infty}\mathrm{d}t'\int_V\mathrm{d}^3r'\,\chi_{BA}(\mathbf{r},\mathbf{r}';t-t')a(\mathbf{r}',t'), \tag{420}$$

where we extended the lower integration limit to $-\infty$, knowing that the perturbation is actually $a(\mathbf{r},t)\Theta(t-t_0)$, and $\Theta(t-t')$ in (419) allows to extend the upper integration limit to ∞.

- The Θ-function in (419) ensures causality, i.e., that the response should follow after the perturbation.

[1] No operator hats in this lecture. It should be clear from the context which quantities are operators and which ones are "c numbers" or functions.

- Note that in (420) the variation in B is written as a convolution of the perturbation a and the linear response function, which only depends on unperturbed entities.

- For translationally invariant systems, the response function can only depend on $\mathbf{r}-\mathbf{r}'$,

$$\chi_{BA}(\mathbf{r},\mathbf{r}';t-t') = \chi_{BA}(\mathbf{r}-\mathbf{r}';t-t'). \tag{421}$$

- Writing

$$f(\mathbf{r},t) = \frac{1}{V}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\, f_{\mathbf{k}}(\omega)\ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{422}$$

$$f_{\mathbf{k}}(\omega) = \int_{-\infty}^{\infty}\mathrm{d}t\int_V \mathrm{d}^3 r\, f(\mathbf{r},t)\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{423}$$

  for all time and space-dependent quantities a, B, and χ_{BA}, we have, using (421),

$$\begin{aligned}
&\frac{1}{V}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\delta B_{\mathbf{k}}(\omega)\ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\
&= \int_{-\infty}^{\infty}\mathrm{d}t'\int \mathrm{d}^3 r'\,\frac{1}{V}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\chi_{BA,\mathbf{k}}(\omega)\ \mathrm{e}^{\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\,\frac{1}{V}\sum_{\mathbf{k}'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\,a_{\mathbf{k}'}(\omega')\ \mathrm{e}^{\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}'-\omega' t')}\\
&= \frac{1}{2\pi V}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\chi_{BA,\mathbf{k}}(\omega)\ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}a_{\mathbf{k}}(\omega),
\end{aligned}$$

  and thus

$$\boxed{\delta B_{\mathbf{k}}(\omega) = \chi_{BA,\mathbf{k}}(\omega)a_{\mathbf{k}}(\omega)}\ , \tag{424}$$

  which is nothing but the convolution theorem applied to (420).

- For the linear response function in $(\omega,\mathbf{k})$-space we obtain

$$\chi_{BA,\mathbf{k}}(\omega) = \int_{-\infty}^{\infty}\mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\,\frac{\mathrm{i}}{\hbar}\Theta(t-t')\langle\psi_0|[B(\mathbf{r},t-t'),A(\mathbf{r}')]\,|\psi_0\rangle\ \mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]},$$

  i.e.

$$\chi_{BA,\mathbf{k}}(\omega) = \frac{\mathrm{i}}{\hbar}\int_0^{\infty}\mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\,\langle\psi_0|[B(\mathbf{r},t-t'),A(\mathbf{r}')]\,|\psi_0\rangle\ \mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}. \tag{425}$$

- It is interesting to see that the linear response is determined by the commutator between the observable of interest and the operator to which the perturbation couples in the Hamiltonian.

- We know from the Statistical Physics lecture that in the case of non-pure states, i.e., mixtures, the expectation value $\langle\psi|\cdot|\psi\rangle$ is to be replaced by the corresponding trace $\mathrm{Tr}\,(\rho\,\cdot)$ where ρ is the statistical operator. Hence

$$\boxed{\chi_{BA,\mathbf{k}}(\omega)=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\,\mathrm{Tr}\left(\rho[B(\mathbf{r},t-t'),A(\mathbf{r}')]\right)\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}}\,. \tag{426}$$

- Let E_n be the eigenstates of the unperturbed system, $H_0\,|n\rangle=E_n\,|n\rangle$ so that (426) reads in energy representation

$$\chi_{BA,\mathbf{k}}(\omega)=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\sum_n \rho(E_n)\langle n|[B(\mathbf{r},t-t'),A(\mathbf{r}')]\,|n\rangle\;\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}.$$

- Using

$$B(\mathbf{r},t-t')=\mathrm{e}^{\mathrm{i}H_0(t-t')/\hbar}B(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}H_0(t-t')/\hbar}$$

we find

$$\begin{aligned}
\chi_{BA,\mathbf{k}}(\omega) &= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\;\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\\
&\qquad\times\sum_n \rho(E_n)\Big\{\langle n|\,\mathrm{e}^{\mathrm{i}H_0(t-t')/\hbar}B(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}H_0(t-t')/\hbar}A(\mathbf{r}')\,|n\rangle\\
&\qquad\qquad -\langle n|A(\mathbf{r}')\,\mathrm{e}^{\mathrm{i}H_0(t-t')/\hbar}B(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}H_0(t-t')/\hbar}\,|n\rangle\Big\}\\
&= \frac{\mathrm{i}}{\hbar V^2}\sum_{\mathbf{k}'}\sum_{\mathbf{k}''}\int_0^\infty \mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\;\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\,\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot\mathbf{r}}\,\mathrm{e}^{\mathrm{i}\mathbf{k}''\cdot\mathbf{r}'}\\
&\qquad\times\sum_{nm}\rho(E_n)\Big\{\langle n|\,\mathrm{e}^{\mathrm{i}H_0(t-t')/\hbar}B_{\mathbf{k}'}\,|m\rangle\,\langle m|\,\mathrm{e}^{-\mathrm{i}H_0(t-t')/\hbar}A_{\mathbf{k}''}\,|n\rangle\\
&\qquad\qquad -\langle n|A_{\mathbf{k}''}\,\mathrm{e}^{\mathrm{i}H_0(t-t')/\hbar}\,|m\rangle\,\langle m|B_{\mathbf{k}'}\,\mathrm{e}^{-\mathrm{i}H_0(t-t')/\hbar}\,|n\rangle\Big\}\\
&= \frac{\mathrm{i}}{\hbar V^2}\sum_{\mathbf{k}'}\sum_{\mathbf{k}''}\int_0^\infty \mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\\
&\qquad\times\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\,\mathrm{e}^{\frac{\mathrm{i}}{2}[(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{k}'-\mathbf{k}'')+(\mathbf{r}+\mathbf{r}')\cdot(\mathbf{k}'+\mathbf{k}'')]}\\
&\qquad\times\sum_{nm}\rho(E_n)\Big\{\mathrm{e}^{-\mathrm{i}\omega_{mn}(t-t')}(B_{\mathbf{k}'})_{nm}(A_{\mathbf{k}''})_{mn}-\mathrm{e}^{\mathrm{i}\omega_{mn}(t-t')}(A_{\mathbf{k}''})_{nm}(B_{\mathbf{k}'})_{mn}\Big\}
\end{aligned}$$

Here $\hbar\omega_{mn}=E_m-E_n$ and matrix elements like $\langle m|B_{\mathbf{k}}\,|n\rangle$ are written as $(B_{\mathbf{k}})_{mn}$.

Integrating both sides over $(r+r')/2$ yields a V on the left hand side and a $V\delta(\mathbf{k}', -\mathbf{k}'')$ on the right hand side so that

$$\begin{aligned}
\chi_{BA,\mathbf{k}}(\omega) &= \frac{\mathrm{i}}{\hbar V^2}\sum_{\mathbf{k}'}\int_0^\infty \mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\ \mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\,\mathrm{e}^{\mathrm{i}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{k}'} \\
&\quad\times\sum_{nm}\rho(E_n)\Big\{\mathrm{e}^{-\mathrm{i}\omega_{mn}(t-t')}(B_{\mathbf{k}'})_{nm}(A_{-\mathbf{k}'})_{mn} - \mathrm{e}^{\mathrm{i}\omega_{mn}(t-t')}(A_{-\mathbf{k}'})_{nm}(B_{\mathbf{k}'})_{mn}\Big\} \\
&= \frac{\mathrm{i}}{\hbar V}\int_0^\infty \mathrm{d}(t-t')\ \sum_{nm}\rho(E_n)\Big\{\mathrm{e}^{-\mathrm{i}(\omega_{mn}-\omega)(t-t')}(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn} \\
&\qquad - \mathrm{e}^{\mathrm{i}(\omega_{mn}+\omega)(t-t')}(A_{-\mathbf{k}})_{nm}(B_{\mathbf{k}})_{mn}\Big\}
\end{aligned}$$

- We perform the time integration by adding a small imaginary part to ω, i.e., $\omega\to\omega+\mathrm{i}\eta$,

$$\begin{aligned}
\chi_{BA,\mathbf{k}}(\omega) &= \frac{\mathrm{i}}{\hbar V}\lim_{\eta\to0^+}\sum_{nm}\rho(E_n)\left\{\frac{-(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}}{-\mathrm{i}(\omega_{mn}-\omega-\mathrm{i}\eta)} - \frac{-(A_{-\mathbf{k}})_{nm}(B_{\mathbf{k}})_{mn}}{\mathrm{i}(\omega_{mn}+\omega+\mathrm{i}\eta)}\right\} \\
&= -\frac{1}{\hbar V}\lim_{\eta\to0^+}\sum_{nm}\rho(E_n)\left\{\frac{(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}}{\omega+\mathrm{i}\eta-\omega_{mn}} - \frac{(A_{-\mathbf{k}})_{nm}(B_{\mathbf{k}})_{mn}}{\omega+\mathrm{i}\eta+\omega_{mn}}\right\} \\
&= -\frac{1}{\hbar V}\lim_{\eta\to0^+}\sum_{nm}\left\{\rho(E_n)\frac{(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}}{\omega+\mathrm{i}\eta-\omega_{mn}} - \rho(E_m)\frac{(A_{-\mathbf{k}})_{mn}(B_{\mathbf{k}})_{nm}}{\omega+\mathrm{i}\eta-\omega_{mn}}\right\} \\
&= -\frac{1}{\hbar V}\lim_{\eta\to0^+}\sum_{nm}\Big\{\rho(E_n)-\rho(E_m)\Big\}\frac{(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}}{\omega+\mathrm{i}\eta-\omega_{mn}}.
\end{aligned}$$

In the second last step we relabelled indices and used $\omega_{mn}=-\omega_{nm}$.

- With, e.g., the canonical density operator

$$\rho(E_n) = \frac{\mathrm{e}^{-E_n/(k_\mathrm{B}T)}}{Z}, \qquad Z=\mathrm{Tr}\ \mathrm{e}^{-H_0/(k_\mathrm{B}T)}=\sum_n \mathrm{e}^{-E_n/(k_\mathrm{B}T)}$$

we have

$$\boxed{\chi_{BA,\mathbf{k}}(\omega) = -\frac{1}{\hbar V}\lim_{\eta\to0^+}\sum_{nm}\rho(E_n)\left(1-\ \mathrm{e}^{-\hbar\omega_{mn}/(k_\mathrm{B}T)}\right)\frac{(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}}{\omega+\mathrm{i}\eta-\omega_{mn}}}. \tag{427}$$

- We see that

$$\chi_{AB,-\mathbf{k}}(-\omega) = -\frac{1}{\hbar V}\lim_{\eta\to0^+}\sum_{nm}\Big\{\rho(E_n)-\rho(E_m)\Big\}\frac{(A_{-\mathbf{k}})_{nm}(B_{\mathbf{k}})_{mn}}{-\omega+\mathrm{i}\eta-\omega_{mn}}$$

$$= -\frac{1}{\hbar V}\lim_{\eta\to 0^+}\sum_{nm}\Big\{\rho(E_m)-\rho(E_n)\Big\}\frac{(A_{-\mathbf{k}})_{mn}(B_{\mathbf{k}})_{nm}}{-\omega+\mathrm{i}\eta+\omega_{mn}}$$
$$= -\frac{1}{\hbar V}\lim_{\eta\to 0^+}\sum_{nm}\Big\{\rho(E_n)-\rho(E_m)\Big\}\frac{(A_{-\mathbf{k}})_{mn}(B_{\mathbf{k}})_{nm}}{\omega-\mathrm{i}\eta-\omega_{mn}}$$

and thus

$$\chi_{AB,-\mathbf{k}}(-\omega) = [\chi_{BA,\mathbf{k}}(\omega)]^*. \tag{428}$$

- Moreover, we find

$$-\frac{\mathrm{i}}{2}[\chi_{BA,\mathbf{k}}(\omega)-\chi_{AB,-\mathbf{k}}(-\omega)]$$
$$= \frac{\mathrm{i}}{2\hbar V}\lim_{\eta\to 0^+}\sum_{nm}[\rho(E_n)-\rho(E_m)]\left[\frac{(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}}{\omega+\mathrm{i}\eta-\omega_{mn}}-\frac{(A_{-\mathbf{k}})_{nm}(B_{\mathbf{k}})_{mn}}{-\omega+\mathrm{i}\eta-\omega_{mn}}\right]$$
$$= \frac{\mathrm{i}}{2\hbar V}\lim_{\eta\to 0^+}\sum_{nm}[\rho(E_n)-\rho(E_m)](B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}\left[\frac{1}{\omega+\mathrm{i}\eta-\omega_{mn}}+\frac{1}{-\omega+\mathrm{i}\eta+\omega_{mn}}\right]$$
$$= \frac{1}{\hbar V}\sum_{nm}[\rho(E_n)-\rho(E_m)](B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}\underbrace{\lim_{\eta\to 0^+}\frac{\eta}{(\omega-\omega_{mn})^2+\eta^2}}_{\pi\delta(\omega-\omega_{mn})}$$
$$= \frac{\pi}{\hbar V}\left(1-\mathrm{e}^{-\hbar\omega/(k_{\mathrm{B}}T)}\right)\sum_{nm}\rho(E_n)\,(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}\;\delta(\omega-\omega_{mn}), \tag{429}$$

which we will need in the next subsection.

7.2.2 *Generalized dynamic structure factor*

- Consider instead of (426) the corresponding expression with an *anti-commutator* $[B,A]_+ = BA + AB$ instead of a commutator, the lower limit for the time integration $-\infty$, and a modified prefactor,

$$\boxed{S_{BA,\mathbf{k}}(\omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\,\mathrm{Tr}\,\left\{\rho[B(\mathbf{r},t-t'),A(\mathbf{r}')]_+\right\}\,\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}}. \tag{430}$$

- Clearly, (410) is a special case of this expression, with both A and B density fluctuations.

E Show that an analogous calculation to the one in the previous subsection yields

$$\boxed{S_{BA,\mathbf{k}}(\omega) = \frac{1}{2V}\left(1+\mathrm{e}^{-\hbar\omega/(k_{\mathrm{B}}T)}\right)\sum_{nm}\rho(E_n)\,(B_{\mathbf{k}})_{nm}(A_{-\mathbf{k}})_{mn}\,\delta(\omega-\omega_{mn})}. \tag{431}$$

- Comparison with (429) shows that

$$S_{BA,\mathbf{k}}(\omega) \;=\; \frac{-\frac{1}{2V}\left(1+\mathrm{e}^{-\hbar\omega/(k_\mathrm{B}T)}\right)\frac{\mathrm{i}}{2}[\chi_{BA,\mathbf{k}}(\omega)-\chi_{AB,-\mathbf{k}}(-\omega)]}{\frac{\pi}{\hbar V}\left(1-\mathrm{e}^{-\hbar\omega/(k_\mathrm{B}T)}\right)}$$

and therefore

$$\boxed{S_{BA,\mathbf{k}}(\omega) = -\frac{\mathrm{i}\hbar}{4\pi}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right)\left[\chi_{BA,\mathbf{k}}(\omega)-\chi_{AB,-\mathbf{k}}(-\omega)\right]}\,. \tag{432}$$

This is the *fluctuation-dissipation theorem*.

- The dynamic structure factor is real. In fact, with (428) the fluctuation-dissipation theorem (432) can be written as

$$S_{BA,\mathbf{k}}(\omega) = \frac{\hbar}{2\pi}\coth\left(\frac{\hbar\omega}{2k_\mathrm{B}T}\right)\,\mathrm{Im}\,\chi_{BA,\mathbf{k}}(\omega). \tag{433}$$

- The fluctuation-dissipation theorem thus states that the power spectrum of the *fluctuations* of a system in thermal equilibrium is directly related to the imaginary part of the linear response function. The imaginary part of the linear response function describes how a perturbation is *dissipated* by the system; hence the name "fluctuation-dissipation theorem".

- If $k_\mathrm{B}T \gg \hbar\omega$ the classical version of the fluctuation-dissipation theorem

$$S_{BA,\mathbf{k}}(\omega) = \frac{k_\mathrm{B}T}{\pi\omega}\,\mathrm{Im}\,\chi_{BA,\mathbf{k}}(\omega) \tag{434}$$

results (without $\hbar$ inside, of course).

7.3 STRUCTURE FACTOR OF EQUILIBRIUM PLASMA

- In order to bring our general expression (430)

$$S_{BA,\mathbf{k}}(\omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\,\mathrm{Tr}\,\left\{\rho[B(\mathbf{r},t-t'),A(\mathbf{r}')]_+\right\}\,\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}$$

into the form (410)

$$S(\mathbf{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\,\left\langle \delta n_\mathbf{k}(t)\,\delta n_{-\mathbf{k}}(0)\right\rangle\,\mathrm{e}^{\mathrm{i}\omega t}$$

we try

$$A(\mathbf{r},t) = B(\mathbf{r},t) = \delta n(\mathbf{r},t)$$

so that (418) is

$$H'(t) = -\int \mathrm{d}^3 r\, \delta n(\mathbf{r},t)\, q\phi_{\text{ext}}(\mathbf{r},t) \tag{435}$$

with $a(\mathbf{r},t) = q\phi_{\text{ext}}(\mathbf{r},t)$ the perturbating external potential that couples to the density and causes the response of the system.

- Indeed, we have

$$\begin{aligned}
S_{\delta n\delta n,\mathbf{k}}(\omega) &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}(t-t')\int_V \mathrm{d}^3(r-r')\,\mathrm{Tr}\,\left\{\rho[\delta n(\mathbf{r},t-t'),\delta n(\mathbf{r}')]_+\right\}\,\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\int_V \mathrm{d}^3(r-r')\left\langle \delta n(\mathbf{r},t),\delta n(\mathbf{r}')\right\rangle\,\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega t]}\\
&= \frac{1}{2\pi V^2}\sum_{\mathbf{k}',\mathbf{k}''}\int_{-\infty}^{\infty}\mathrm{d}t\int_V \mathrm{d}^3(r-r')\left\langle \delta n_{\mathbf{k}'}(t)\delta n_{\mathbf{k}''}\right\rangle\,\mathrm{e}^{\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}+\mathbf{k}''\cdot\mathbf{r}')}\,\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega t]}\\
S_{\delta n\delta n,\mathbf{k}}(\omega)V &= \frac{1}{2\pi V^2}\sum_{\mathbf{k}',\mathbf{k}''}\int_{-\infty}^{\infty}\mathrm{d}t\int_V \mathrm{d}^3(r+r')/2\int_V \mathrm{d}^3(r-r')\left\langle \delta n_{\mathbf{k}'}(t)\delta n_{\mathbf{k}''}\right\rangle\,\mathrm{e}^{\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}+\mathbf{k}''\cdot\mathbf{r}')}\,\mathrm{e}^{-\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega t]}\\
&= \frac{1}{2\pi V^2}\sum_{\mathbf{k}',\mathbf{k}''}\int_{-\infty}^{\infty}\mathrm{d}t\int_V \mathrm{d}^3 r\int_V \mathrm{d}^3 r'\left\langle \delta n_{\mathbf{k}'}(t)\delta n_{\mathbf{k}''}\right\rangle\,\mathrm{e}^{-\mathrm{i}[\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')-\mathbf{r}'\cdot(\mathbf{k}''+\mathbf{k})-\omega t]}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\left\langle \delta n_{\mathbf{k}}(t)\delta n_{-\mathbf{k}}\right\rangle\,\mathrm{e}^{\mathrm{i}\omega t}.
\end{aligned}$$

Therefore

$$S(\mathbf{k},\omega) = V S_{\delta n\delta n,\mathbf{k}}(\omega), \tag{436}$$

i.e. our generalized dynamic structure factor $S_{\delta n\delta n,\mathbf{k}}(\omega)$ has been defined *per volume* and the original $S(\mathbf{k},\omega)$ in eq. (410) as an extensive quantity scaling *with volume*. The difference is just a matter of definition, of course.

- The linear response function is determined via (424),

$$\delta n_{\mathbf{k}}(\omega) = \chi_{\delta n\delta n}(\mathbf{k},\omega)\, q\phi_{\text{ext}}(\mathbf{k},\omega) \qquad \Rightarrow \qquad \chi_{\delta n\delta n}(\mathbf{k},\omega) = \frac{\delta n_{\mathbf{k}}(\omega)}{q\phi_{\text{ext}}(\mathbf{k},\omega)}.$$

- The induced potential because of the density response is determined via Poisson's equation

$$\epsilon_0\nabla^2\phi_{\text{ind}}(\mathbf{r},t) = q\delta n(\mathbf{r},t) \qquad \Rightarrow \qquad \phi_{\text{ind}}(\mathbf{k},\omega) = -\frac{q}{k^2\epsilon_0}\,\delta n_{\mathbf{k}}(\omega)$$

so that

$$\phi_{\text{ind}}(\mathbf{k},\omega) = -\frac{q^2}{k^2\epsilon_0}\,\chi_{\delta n\delta n}(\mathbf{k},\omega)\phi_{\text{ext}}(\mathbf{k},\omega).$$

- The total potential is

$$\phi_{\text{tot}}(\mathbf{k},\omega) = \phi_{\text{ind}}(\mathbf{k},\omega) + \phi_{\text{ext}}(\mathbf{k},\omega) = \left[1 - \frac{q^2}{k^2\epsilon_0}\,\chi_{\delta n\delta n}(\mathbf{k},\omega)\right]\phi_{\text{ext}}(\mathbf{k},\omega). \tag{437}$$

- On the other hand,

$$\phi_{\text{tot}}(\mathbf{k},\omega) = \frac{1}{\epsilon(\mathbf{k},\omega)}\,\phi_{\text{ext}}(\mathbf{k},\omega) \tag{438}$$

where $\epsilon(\mathbf{k},\omega)$ is the dielectric function (previously known as $D(\mathbf{k},\omega)$ in section 3.3.1).

E Show (438), starting from the definition of $\epsilon(\mathbf{k},\omega)$ (as learned in *Electrodynamics*).

- Comparison of (437) and (438) yields

$$\chi_{\delta n\delta n}(\mathbf{k},\omega) = \frac{k^2\epsilon_0}{q^2}\left[1 - \frac{1}{\epsilon(\mathbf{k},\omega)}\right]. \tag{439}$$

- With (434) and (436) we find

$$S(\mathbf{k},\omega) = V\frac{k_{\text{B}}T}{\pi\omega}\operatorname{Im}\chi_{\delta n\delta n,\mathbf{k}}(\omega) = -\frac{k^2\epsilon_0}{q^2}V\frac{k_{\text{B}}T}{\pi\omega}\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega)} = -\frac{k^2\lambda_{\text{D}}^2}{\pi\omega}N\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega)}$$

$$\Rightarrow \qquad \boxed{S(\mathbf{k},\omega) = -\frac{Nk^2}{\pi\omega k_{\text{D}}^2}\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega)}} \tag{440}$$

where $k_{\text{D}} = \lambda_{\text{D}}^{-1}$ (not $2\pi\lambda_{\text{D}}^{-1}$).

- Equation (440) establishes a connection between $S(\mathbf{k},\omega)$ and $\operatorname{Im}\epsilon(\mathbf{k},\omega)$. As we know how to calculate $\epsilon(\mathbf{k},\omega)$ (at least in principle) we now know how to calculate $S(\mathbf{k},\omega)$.

- Using (412) we determine the static structure factor by integrating over all frequencies,

$$S(\mathbf{k}) = -\frac{k^2}{\pi k_{\text{D}}^2}\int\frac{\mathrm{d}\omega}{\omega}\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega)}.$$

- With the help of the third of the following KRAMERS-KRONIG relations (evaluated for $\omega = 0$)

$$\operatorname{Re}\epsilon(\mathbf{k},\omega) - 1 = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\operatorname{Im}\epsilon(\mathbf{k},\omega')}{\omega' - \omega} \tag{441}$$

$$\operatorname{Im}\epsilon(\mathbf{k},\omega) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{1 - \operatorname{Re}\epsilon(\mathbf{k},\omega')}{\omega' - \omega} \tag{442}$$

$$\operatorname{Re}\frac{1}{\epsilon(\mathbf{k},\omega)} - 1 = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega')}}{\omega' - \omega} \tag{443}$$

$$\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega)} = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{1 - \operatorname{Re}\frac{1}{\epsilon(\mathbf{k},\omega')}}{\omega' - \omega} \tag{444}$$

one obtains

$$S(\mathbf{k}) = -\frac{k^2}{\pi k_{\mathrm{D}}^2}\int \frac{\mathrm{d}\omega'}{\omega'}\operatorname{Im}\frac{1}{\epsilon(\mathbf{k},\omega')} = \frac{k^2}{k_{\mathrm{D}}^2}\left[1 - \operatorname{Re}\frac{1}{\epsilon(\mathbf{k},0)}\right]$$

and with $\operatorname{Re}\epsilon(\mathbf{k},0) \simeq 1 + \frac{k_{\mathrm{D}}^2}{k^2}$

$$\boxed{S(\mathbf{k}) = \frac{k^2}{k^2 + k_{\mathrm{D}}^2}} \,. \tag{445}$$

- The pair correlation function $p(\mathbf{r} - \mathbf{r}') = \int \mathrm{d}^3v' \int \mathrm{d}^3v \, G(X, X')$ can be calculated using (408), by FOURIER-transforming back

$$\frac{1}{n}[S(\mathbf{k}) - 1] = \int_V \mathrm{d}^3r \; p(\mathbf{r}) \; \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

E Show that in that way

$$p(\mathbf{r}) = -\frac{k_{\mathrm{D}}^2}{4\pi n r} \, \mathrm{e}^{-k_{\mathrm{D}} r} \tag{446}$$

results.

- Writing this as

$$p(\mathbf{r}) = -\frac{\lambda_{\mathrm{D}}}{3\frac{4}{3}\pi\lambda_{\mathrm{D}}^3 n \, r} \, \mathrm{e}^{-r/\lambda_{\mathrm{D}}} = -\frac{1}{3N_{\mathrm{D}}} \frac{\lambda_{\mathrm{D}}}{r} \, \mathrm{e}^{-r/\lambda_{\mathrm{D}}} \tag{447}$$

we see that the reduction (due to repulsion of like-charge particles) of the probability $p(\mathbf{r}) + 1$ to find a particle at $\mathbf{r}$ if another one is at the origin leads to a functional dependence we know from the treatment of DEBYE screening in section 1.1. We also see that our approximation for the dielectric function breaks down for too small r because $p+1$ becomes negative, which is nonsensical for a probability.

E Show that the energy density in (409) for $p(\mathbf{r})$ according (447) and a COULOMB potential $v(\mathbf{r}) = q^2/(4\pi\epsilon_0 r)$ reads

$$\frac{\langle H \rangle}{V} = \frac{3}{2} n k_\mathrm{B} T \left[1 - \frac{1}{9N_\mathrm{D}} \right] . \tag{448}$$

- The term $\sim N_\mathrm{D}^{-1}$ is the correlation energy density due to the DEBYE-screened COULOMB interaction. It is a correction in first order N_D^{-1} to the $N_\mathrm{D}^{-1} \to 0$ ideal-plasma result $\frac{\langle H \rangle}{V} = \frac{3}{2} n k_\mathrm{B} T$.